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New Exact Solutions for Coupled Korteweg-de Varies-Zakharov-Kuznetsov Equation and Modified Korteweg-de Varies-Zakharov-Kuznetsov Equation

M. S. AL-Amry[1]
Dept. of Math. Faculty of Education, Aden Univ.

A. S. Mohammed[2]
Dept. of Math. Faculty of Education, Aden Univ.

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Abstract: In this paper, we present a model of a coupled of Korteweg-de Varies-Zakharov-Kuznetsov (KdV-ZK) equation and a modified Korteweg-de Varies-Zakharov-Kuznetsov (mKdv-ZK) equation, we apply the mapping method to solve the equation. Exact travelling wave solutions are obtained and expressed in terms of hyperbolic functions, trigonometric functions and rational functions.

Keywords: The (Kdv-ZK) equation, the(mKdv-ZK) equation, the c(kdv-zk and mKdv-ZK) equation, exact solution and the mapping method.

1. Introduction: In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed.

These equations appear in condensed matter, solid state physics, fluid mechanics, chemical kinetics, plasma physics, nonlinear optics, propagation of fluxions in Josephson junctions, theory of turbulence, ocean dynamics, biophysics star formation, and many others.

A variety of useful methods, Example of the methods that have been used so far are:

The Extended Hyperbolic function method [1], the First Integral method [2,4], the Sine-cosine method [3], the Algebraic method [5], an improve F-expansion method [6], variational relatively method [7], tanh-coth method [8], Jacobi elliptic function expansion method [9], the mapping method [10], the generalized riccati equation mapping method [11], simplest equation method [12], the $\left(\frac{G'}{G^2}\right)$-Expansion method [13], tan-cot method [14] and many other, have been proposed to obtain exact solutions.

With the availability of symbolic computation packages like Maple or Mathematica, the search for obtaining exact solutions of nonlinear partial differential equations (nPDEs) has become more and more stimulating for mathematicians and scientists. Having exact solutions of nPDEs makes it possible to study nonlinear physical phenomena thoroughly and facilitates testing the numerical solvers as well as aiding the stability analysis of solutions.

In Sec. 3 of this paper, we use the mapping method [10] to find some new solutions of the c(kdv-zk and mKdv–ZK) equation [15],[17].

2. Description the mapping method: This method was firstly proposed by Peng [12], in 2003 as follows, for a given nonlinear partial differential equation, say, in two variables, Consider the general nonlinear partial differential equations (nPDEs), say, in two variables,

$$P(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0, \tag{1}$$

Will be simply reviewed as follows:

Step 1: Use the wave variable $\xi = \lambda(x - ct)$ to change the nPDE in to nODE:

$$Q(u\,, u', u'', \dots) = 0\,, \tag{2}$$

Where Q is a polynomial of $u(\xi)$ and its total derivatives $u'(\xi), u''(\xi), \dots$

Step 2: We suppose that the solution of Eq. (2) has the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{n} a_i(f(\xi))^i, \tag{3}$$

where the coefficients $a_i(i = 0,1,2, \dots)$, are constants to be determined, and $f = f(\xi)$ satisfies a nonlinear ordinary differential equation

$$\frac{df(\xi)}{d\xi} = \sqrt{pf^2(\xi) + \frac{1}{2}\,qf^4(\xi) + r} \tag{4}$$

where $q\,,p\,,r$ are constants to be determined later.

In (2008), A. Elgarayhi in [16], used the same solution formula in Eq. (3), but $f(\xi)$ satisfies the auxiliary equation

$$\frac{df(\xi)}{d\xi} = \sqrt{pf^2(\xi) + \frac{1}{2}\,qf^4(\xi) + \frac{1}{3}\,sf^6(\xi) + r} \tag{4}$$

Step 3: We determine the positive integer n in Eq. (3) by balancing the highest-order derivatives and the highest-order nonlinear terms in Eq. (2)

Step 4: Substituting Eq.(3) along with Eq.(4.1) into Eq.(2) and collecting all the coefficients of $f^i(\xi)$, then setting them to zero, yield a set of algebraic equations.

Step 5: Solving the algebraic equations in step 4, using the Maple to find $a_i\,,q\,,p\,,r\ and\ c$

Step 6: The Eq. (4) has many solutions as described in the following:

1) $f(\xi) = sech(\xi), [p = 1,\ q = -2,\ r = 0]$,
2) $f(\xi) = tanh(\xi), [p = -2,\ q = 2,\ r = 1]$,
3) $f(\xi) = \frac{1}{2}tanh(2\xi)\ or\ \frac{1}{2}coth(2\xi), [p = -8,\ q = 32,\ r = 1]$,
4) $f(\xi) = \frac{1}{2}tan(2\xi)\ or\ \frac{1}{2}cot(2\xi), [p = 8,\ q = 32,\ r = 1]$,
5) $f(\xi) = sn(\xi)\,, [p = -(m^2 + 1),\ q = 2m^2,\ r = 1]$,
6) $f(\xi) = ns(\xi)\,, [p = -(m^2 + 1),\ q = 2,\ r = m^2]$,
7) $f(\xi) = cd(\xi)\,, [p = -(m^2 + 1),\ q = 2m^2\,, r = 1]$,
8) $f(\xi) = dc(\xi)\,, [p = -(m^2 + 1),\ q = 2,\ r = m^2]$,
9) $f(\xi) = cn(\xi)\,, [p = 2m^2 - 1,\ q = -2m^2,\ r = 1 - m^2]$,
10) $f(\xi) = nc(\xi)\,, [p = 2m^2 - 1,\ q = 2(1 - m^2),\ r = -m^2]$,
11) $f(\xi) = dn(\xi)\,, [p = 2 - m^2,\ q = -2,\ r = -(1 - m^2)]$,
12) $f(\xi) = nd(\xi)\,, [p = 2 - m^2,\ q = 2(m^2 - 1),\ r = -1]$,
13) $f(\xi) = cs(\xi)\,, [p = 2-m^2,\ q = 2,\ r = 1 - m^2]$,
14) $f(\xi) = sc(\xi)\,, [p = 2-m^2,\ q = 2(1 - m^2),\ r = 1]$,
15) $f(\xi) = ds(\xi)\,, [p = -1+2m^2,\ q = 2,\ r = -m^2(1 - m^2)]$,
16) $f(\xi) = sd(\xi)\,, [p = -1+2m^2,\ q = 2m^2(m^2 - 1),\ r = 1]$,
17) $f(\xi) = sc(\xi) \pm nc(\xi), \left[p = \frac{1+m^2}{2},\ q = \frac{1-m^2}{2},\ r = \frac{1-m^2}{4}\right]$,
18) $f(\xi) = \frac{sn(\xi)}{1 \pm dn(\xi)}\,, \left[p = \frac{m^2-2}{2},\ q = \frac{m^2}{2},\ r = \frac{1}{4}\right]$,
19) $f(\xi) = \frac{dn(\xi)}{1 \pm m\,sn(\xi)}\,, \left[p = \frac{m^2+1}{2},\ q = \frac{m^2-1}{2},\ r = \frac{1-m^2}{4}\right]$,
20) $f(\xi) = m\,cn(\xi) \pm dn(\xi), \left[p = \frac{m^2+1}{2},\ q = \frac{-1}{2},\ r = \frac{-(1-m^2)^2}{4}\right]$,

21) $f(\xi) = \frac{cn(\xi)}{1 \pm sn(\xi)}$, $\left[p = \frac{m^2+1}{2},\ q = \frac{1-m^2}{2},\ r = \frac{1-m^2}{4}\right]$,

22) $f(\xi) = m\, sn(\xi) \pm i\, dn(\xi)$, $\left[p = \frac{1-2m^2}{2},\ q = \frac{1}{2},\ r = \frac{m^2}{4}\right]$,

23) $f(\xi) = m\, sn(\xi) \pm i\, cn(\xi)$, $\left[p = \frac{m^2-2}{2},\ q = \frac{m^2}{2},\ r = \frac{m^2}{4}\right]$,

24) $f(\xi) = ns(\xi) \pm ds(\xi)$, $\left[p = \frac{m^2-2}{2},\ q = \frac{1}{2},\ r = \frac{m^4}{4}\right]$,

25) $f(\xi) = ns(\xi) \pm cs(\xi)$, $\left[p = \frac{1-2m^2}{2},\ q = \frac{1}{2},\ r = \frac{1}{4}\right]$,

26) $f(\xi) = \frac{cn(\xi)}{\sqrt{1-m^2}\, sn(\xi) \pm dn(\xi)}$, , $\left[p = \frac{1-2m^2}{2},\ q = \frac{1}{2},\ r = \frac{1}{4}\right]$,

27) $f(\xi) = \frac{sn(\xi)}{cn(\xi) \pm dn(\xi)}$, $\left[p = \frac{1+m^2}{2},\ q = \frac{(1-m^2)^2}{2},\ r = \frac{1}{4}\right]$,

28) $f(\xi) = \frac{cn(\xi)}{\sqrt{1-m^2} \pm dn(\xi)}$, $\left[p = \frac{m^2-2}{2},\ q = \frac{m^2}{2},\ r = \frac{1}{4}\right]$,

29) $f(\xi) = \sqrt{m^2-1}\, sd(\xi) \pm cd(\xi)$, $\left[p = \frac{m^2-2}{2},\ q = \frac{m^2}{2},\ r = \frac{m^2}{4}\right]$,

30) $f(\xi) = m\, cd(\xi) \pm i\sqrt{1-m^2}\, nd(\xi)$, , $\left[p = \frac{1-2m^2}{2},\ q = \frac{1}{2},\ r = \frac{1}{4}\right]$,

31) $f(\xi) = sc(\xi) \pm dc(\xi)$, , $[p = \frac{1-2m^2}{2},\ q = \frac{1}{2},\ r = \frac{1}{4}]$,

32) $f(\xi) = m\, sd(\xi) \pm nd(\xi)$, , $[p = \frac{m^2+1}{2},\ q = \frac{m^2-1}{2},\ r = \frac{m^2-1}{4}]$,

33) $f(\xi) = ds(\xi) \pm cs(\xi)$, , $[p = \frac{m^2+1}{2},\ q = \frac{(1-m^2)^2}{2},\ r = \frac{1}{4}]$,

34) $f(\xi) = cd(\xi) \pm \sqrt{1-m^2}\, nc(\xi)$, , $\left[p = \frac{m^2-2}{2},\ q = \frac{1}{2},\ r = \frac{m^2}{4}\right]$,

35) $f(\xi) = \frac{sn(\xi)dn(\xi)}{cn(\xi)}$, $[p = 2 - 4m^2,\ q = 2,\ r = 1]$,

36) $f(\xi) = \frac{sn(\xi)cn(\xi)}{dn(\xi)}$, $[p = 2m - 4^2,\ q = 2m^4,\ r = 1]$,

37) $f(\xi) = \frac{dn(\xi)cn(\xi)}{sn(\xi)}$, $[p = 2m^2 + 2,\ q = 2,\ r = 1 - 2m^2 + m^4]$,

38) $f(\xi) = \frac{dn(\xi)cn(\xi)}{A(1+sn(\xi))(1+m\, sn(\xi))}$, $\left[p = \frac{m^2+1}{2+3m^2},\ q = \frac{A^2(m^2-1)^2}{2},\ r = \frac{(m^2-1)^2}{4A^2}\right]$,$0 \neq A \in R$

39) $f(\xi) = \frac{dn(\xi)cn(\xi)}{A(1+sn(\xi))(1-m\, sn(\xi))}$, $\left[p = \frac{m^2+1}{2-3m^2},\ q = \frac{A^2(m^2+1)^2}{2},\ r = \frac{(m^2+1)^2}{4A^2}\right]$,$0 \neq A \in R$

40) $f(\xi) = \frac{m\, cn(\xi)\, dn(\xi)}{m\, sn^2(\xi)+1}$, $\left[p = 6m - m^2 - 1, q = \frac{-8}{m}, r = -2m^3 + m^4 + m^2\right]$,

41) $f(\xi) = \frac{m\, cn(\xi)\, dn(\xi)}{m\, sn^2(\xi)-1}$, $\left[p = -6m - m^2 - 1,\ q = \frac{8}{m},\ r = 2m^3 + m^4 + m^2\right]$,

42) $f(\xi) = \frac{m\, sn^2(\xi)-1}{B(m\, sn^2(\xi)+1)}$, $\left[p = 2m^2 + 2,\ q = -2(m^2 + 2m + 1)B^2,\ r = \frac{2m-m^2-1}{B^2}\right]$,

43) $f(\xi) = \frac{m\, sn^2(\xi)+1}{B(m\, sn^2(\xi)-1)}$, $\left[p = 2m^2 + 2,\ q = -2(m^2 - 2m + 1)B^2,\ r = \frac{-(2m+m^2+1)}{B^2}\right]$,

44) $f(\xi) = \sqrt{\frac{-2m^2 p}{(1+m^2)q}}\, sn\left(\sqrt{\frac{-p}{(1+m^2)}}\,\xi\right)$, $\left[r = \frac{2m^2p^2}{(m^2+1)^2 q},\ q > 0, p < 0\right]$,

45) $f(\xi) = \sqrt{\frac{-2p}{(2-m^2)q}}\, dn\left(\sqrt{\frac{p}{(2-m^2)}}\,\xi\right)$, $\left[r = \frac{2(1-m^2)^2p^2}{(m^2-2)^2 q},\ p > 0, q < 0\right]$,

46) $f(\xi) = \sqrt{\frac{-2m^2 p}{(2m^2-1)q}}\, cn\left(\sqrt{\frac{p}{(2m^2-1)}}\,\xi\right)$, $\left[r = \frac{2m^2(m^2-1)^2p^2}{(2m^2-1)^2 q},\ q < 0, p > 0\right]$,

47) $\boldsymbol{f}(\xi) = \frac{-1}{\sqrt{\frac{c}{2}}\xi}$, $[p = 0, q = c, r = 0]$,

48) $f(\xi) = e^{\xi}$, $[p = 1, q = 0, r = 0]$.

Step 7: Substituting the solutions of step 6, into Eq. (3) we have the exact solutions of Eq. (1)

3. New exact solutions for c(Kdv-zk and mKdv-Zk) equation:

The c(Kdv-ZK and mKdv-Zk) equation is given by

$$u_t + \beta\,(u + u^2)u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0,\ \ \beta \neq 0,\quad u = u(x,y,z,t)\,, \tag{5}$$

where

$$u_t + \beta u u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{6}$$

is Korteweg–de Varies–Zakharov–Kuznetsov equation and

$$u_t + \beta u^2 u_x + u_{xxx} + u_{xyy} + u_{zzz} = 0, \tag{7}$$

is a modified Korteweg–de Varies–Zakharov–Kuznetsov equation [15].

Using the transformation $u(x,y,z,t) = u(\xi),\ \xi = \lambda(x + y + z - ct)$ in Eq (1) from (npde) to (node) we get:

$$-c\lambda u' + \lambda\beta(u + u^2)u' + \lambda^3 u''' = 0\,, \tag{8}$$

and integration Eq. (8) we obtain

$$-cu + \frac{\beta}{2}u^2 + \frac{\beta}{3}u^3 + 3\lambda^2 u'' = 0\,, \tag{9}$$

Balancing the highest order of the nonlinear term u^3 with the highest order derivative u'', gives $3k = k + 2$ that gives $k = 1$.Now, we apply the mapping method to solve equation. Consequently, we get the original solutions for our new equation as the following:

Assume, the solution of Eq. (5) has the form

$$u(x,y,z,t) = u(\xi) = a_0 + a_1 f(\xi) + b_1 f^{-1}(\xi), \tag{10}$$

where a_0, a_1 and b_1 are constants.

By substituting Eq. (10) in Eq. (9) and using square Eq. (4.1) and its second derivative, the left hand side is converted into polynomials in $f(\xi)^{\,i},\ (-3 \leq i \leq 3)$. Setting each coefficient of these resulted polynomials to zero, we obtain a set of algebraic equations for $a_0, a_1, b_1, c\ and\ \lambda$. Solving the system of algebraic equations, with the help of algebraic software Maple, we obtain

1) $a_0 = a_0\,, a_1 = 0\,, b_1 = 0\,, \lambda = \lambda\,, c = c\,,$

2) $a_0 = -\frac{1}{2}\,, a_1 = \sqrt{-\frac{1}{4}\frac{q}{p}}\,, b_1 = 0\,, \lambda = \pm\sqrt{\frac{\beta}{36\,p}}\,, c = -\frac{2}{9}\alpha\,,$

3) $a_0 = -\frac{1}{2}\,, a_1 = -\sqrt{-\frac{1}{4}\frac{q}{p}}\,, b_1 = 0\,, \lambda = \pm\sqrt{\frac{\beta}{36\,p}}\,, c = -\frac{2}{9}\alpha.$

The above set of values yields the following exact solutions c(Kdv-ZK and mKdv-ZK) equation. Using Eq. (10), the solution of Eq. (4.1) when $[p = 1,\ q = -2,\ r = 0]$ and the sets of solution (1) -(4), we get:

$u_1(x,y,z,t) = a_0$, a_0 is constant, $for\ \beta > 0\,,$

$$u_{2,3}(x,y,z,t) = -\frac{1}{2} \pm \frac{1}{\sqrt{2}}\ \text{sech}\left(\frac{\sqrt{\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\right),\quad for\ \beta < 0\,,$$

$$u_{4,5}(x,y,z,t) = -\frac{1}{2} \pm \frac{1}{\sqrt{2}}\ \sec\left(\frac{\sqrt{\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\right).$$

Using Eq. (10), the solution of Eq. (4.1) when $[p = -2,\ q = 2,\ r = 1]$ and the sets of solutions (3) -(4), we get: $for\ \beta < 0\,,$

$$u_{6,7}(x,y,z,t) = -\frac{1}{2} \pm \frac{1}{2}\tanh\left(\frac{\sqrt{2\beta}}{12}\left(x + y + z + \frac{1}{6}\beta t\right)\right),\quad for\ \beta > 0\,,$$

$$u_{8,9}(x,y,z,t) = -\frac{1}{2} \pm \frac{i}{2}\tan\left(\frac{\sqrt{2\beta}}{12}\left(x + y + z + \frac{1}{6}\beta t\right)\right).$$

Using Eq. (10), the solution of Eq. (4.1) when $[p = -8,\ q = 32,\ r = 1]$ and the sets of solutions (3) -(4), we get $[u_{6,7}(x,y,z,t),\ u_{8,9}(x,y,z,t)]$ and $for\ \beta < 0$,

$$u_{10,11}(x,y,z,t) = -\frac{1}{2} \pm \frac{1}{2}\coth\left(\frac{\sqrt{2\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right), \quad for\ \beta > 0,$$

$$u_{12,13}(x,y,z,t) = -\frac{1}{2} \pm \frac{i}{2}\cot\left(\frac{\sqrt{2\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right).$$

Using Eq. (10), the solution of Eq. (4.1) when $[p = 8,\ q = 32,\ r = 1]$ and the sets of solutions (3) -(4) we get $[u_{6,7}(x,y,z,t),\ u_{8,9}(x,y,z,t),\ u_{10,11}(x,y,z,t), and\ u_{12,13}(x,y,z,t)]$.

Using Eq. (10), the solution of Eq. (4.1) when $[p = -(m^2+1),\ q = 2m^2,\ r = 1]$ and the sets of solutions (3) -(4), we get $u_{14,\dots,17}(x,y,z,t) = a_0 + a_1 sn(\xi) + b_1 ns(\xi)$.

Note that, when $m \to 1$ we obtain $[\ u_{6,7}(x,y,z,t)\ and\ u_{8,9}(x,y,z,t)]$,when $m \to 0$ we obtain constant solution.

Using Eq. (10), the solution of Eq. (4.1) when $[p = -(m^2+1),\ q = 2,\ r = m^2]$ and the sets of solutions (3) -(4), we get $u_{18,19,\dots,21}(x,y,z,t) = a_0 + a_1 ns(\xi) + b_1 sn(\xi)$.

Note that, when $m \to 1$ We obtain $[\ u_{10,11}(x,y,z,t), u_{12,13}(x,y,z,t)]$, when $m \to 0$, we obtain $for\ \beta < 0$,

$$u_{22,23}(x,y,z,t) = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}\csc\left(\frac{\sqrt{\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right), \quad for\ \beta > 0,$$

$$u_{24,25}(x,y,z,t) = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}\mathrm{csch}\left(\frac{\sqrt{\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right).$$

Using Eq. (10), the solution of Eq. (4.1) when $[p = -(m^2+1),\ q = 2m^2,\ r = 1]$ and the sets of solutions (3) -(4), we get $u_{26,\dots,29}(x,y,z,t) = a_0 + a_1 cd(\xi) + b_1 dc(\xi)$).

Note that, when $m \to 1$ we obtain constant solutions, also when $m \to 0$.

We obtain constant solutions.

Using Eq. (10), the solution of Eq. (4.1) when $[p = -(m^2+1),\ q = 2,\ r = m^2]$ and the sets of solutions (3) -(4), we get $u_{30,\dots,33}(x,y,z,t) = a_0 + a_1 dc(\xi) + b_1 cd(\xi)$.

Note that, when $m \to 1$ we obtain constant solution, when $m \to 0$ we obtain $[u_{2,3}(x,y,z,t), and\ u_{4,5}(x,y,z,t)\ ]$.

Using Eq. (10), the solution of Eq. (4.1) when $[p = 2m^2-1,\ q = -2m^2,\ r = 1-m^2]$ and the sets of solutions (3) -(4), we get $u_{34,\dots,37}(x,y,z,t) = a_0 + a_1 cn(\xi) + b_1 nc(\xi)$.

Note that, when $m \to 1$ we obtain $[\ u_{2,3}(x,y,z,t), and\ u_{4,5}(x,y,z,t)\ ]$, when $m \to 0$ we obtain constant solution.

Using Eq. (10), the solution of Eq. (4.1) when $[p = 2m^2-1,\ q = 2(1-m^2),\ r = -m^2]$ and the sets of solutions (3) -(4), we get $u_{38,\dots,41}(x,y,z,t) = a_0 + a_1 nc(\xi) + b_1 cn(\xi)$.

Note that, when $m \to 1$ we obtain constant solution, when $m \to 0$ we obtain $[\ u_{2,3}(x,y,z,t), and\ u_{4,5}(x,y,z,t)\ ]$.

Using Eq. (10), the solution of Eq. (4.1) when $[p = 2-m^2,\ q = -2,\ r = -(1-m^2)]$ and the sets of solutions (3) -(4), we get $u_{42,\dots,45}(x,y,z,t) = a_0 + a_1 dn(\xi) + b_1 nd(\xi)$.

Note that, when $m \to 1$ we obtain $u_{2,3}(x,y,z,t), and\ u_{4,5}(x,y,z,t)]$, when $m \to 0$ we obtain constant solutions.

Using Eq. (10), the solution of Eq. (4.1) when $[p = 2-m^2,\ q = 2(m^2-1),\ r = -1]$ and the sets of solutions (3) -(4), we get $u_{46,\dots,49}(x,y,z,t) = a_0 + a_1 nd(\xi) + b_1 dn(\xi)$.

Note that, when $m \to 1$ we obtain constant solutions, also when $m \to 0$, we obtain constant solutions.

Using Eq. (10), the solution of Eq. (4.1) when $[p = 2 - m^2,\ q = 2,\ r = 1 - m^2]$ and the sets of solutions (3) -(4), we get $u_{50,\dots,53}(x,y,z,t) = a_0 + a_1 cs(\xi) + b_1 sc(\xi)$.

Note that, when $m \to 1$ we obtain $u_{22,23}(x,y,z,t), u_{24,25}(x,y,z,t)]$, when $m \to 0$ we obtain $[\,u_{10,11}(x,y,z,t), u_{12,13}(x,y,z,t)]$.

Using Eq. (10), the solution of Eq. (4.1) when $[p = 2 - m^2,\ q = 2(1 - m^2),\ r = 1]$ and the sets of solutions (3) -(4), we get $u_{54,\dots,57}(x,y,z,t) = a_0 + a_1 sc(\xi) + b_1 cs(\xi)$.

Note that, when $m \to 1$ we obtain constant solutions, when $m \to 0$, we obtain $[\,u_{6,7}(x,y,z,t), u_{8,9}(x,y,z,t)\,]$.

Using Eq. (10), the solution of Eq. (4.1) when $[p = -1 + 2m^2,\ q = 2,\ r = -m^2(1 - m^2)]$ and the sets of solutions (3) -(4), we get $u_{58,\dots,61}(x,y,z,t) = a_0 + a_1 ds(\xi) + b_1 sd(\xi)$

Note that, when $m \to 1$ we obtain $[\,u_{26,27}(x,y,z,t), and\ \ u_{28,29}(x,y,z,t)\,]$, also when $m \to 0$ we obtain $[\,u_{22,23}(x,y,z,t), and\ \ u_{24,25}(x,y,z,t)\,]$.

Using Eq. (10), the solution of Eq. (4.1) when $[p = -1 + 2m^2,\ q = 2m^2(1 - m^2),\ r = 1]$ and the sets of solutions (3) -(4), we get $u_{62,\dots,65}(x,y,z,t) = a_0 + a_1 sd(\xi) + b_1 ds(\xi)$.

Note that, when $m \to 1$ we obtain constant solutions, also $m \to 0$ we obtain constant solutions.

Using Eq. (10), the solution of Eq. (4.1) when $\left[p = \frac{1+m^2}{2},\ q = \frac{1-m^2}{2},\ r = \frac{1-m^2}{4}\right]$ and the sets of solutions (3) -(4), we get $u_{66,,\dots,69}(x,y,z,t) = a_0 + a_1(sc(\xi) \pm nc(\xi)) + \frac{b_1}{(sc(\xi)\pm nc(\xi))}$

Note that, when $m \to 1$ we obtain constant solutions, when $m \to 0$, we obtain $for\ \ \beta > 0$,

$$u_{70,71}(x,y,z,t) = -\frac{1}{2} + \frac{i}{2}\left(\tan\left(\frac{\sqrt{2\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\right) \pm \sec\left(\frac{\sqrt{2\beta}}{6}\ \left(x + y + z + \frac{1}{6}\beta t\right)\right)\right),$$

$$u_{72,73}(x,y,z,t) = -\frac{1}{2} - \frac{i}{2}\left(\tan\left(\frac{\sqrt{2\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\right) \pm \sec\left(\frac{\sqrt{2\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\right)\right),$$

$for\ \ \beta < 0$,

$$u_{74,75}(x,y,z,t) = -\frac{1}{2} + \frac{i}{2}\left(i\tanh\left(\frac{\sqrt{2\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\right) \pm \operatorname{sech}\left(\frac{\sqrt{2\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\right)\right),$$

$$u_{76,77}(x,y,z,t) = -\frac{1}{2} - \frac{i}{2}\left(i\tanh\left(\frac{\sqrt{2\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\ \right) \pm \operatorname{sech}\left(\frac{\sqrt{2\beta}}{6}\left(x + y + z + \frac{1}{6}\beta t\right)\right)\right).$$

Using Eq. (10), the solution of Eq. (4.1) when $\left[p = \frac{m^2-2}{2},\ q = \frac{m^2}{2},\ r = \frac{1}{4}\right]$ and the sets of solutions (3) -(4) , we get $u_{78,\dots,81}(x,y,z,t) = a_0 + a_1\left(\frac{sn(\xi)}{1\pm dn(\xi)}\right) + b_1\left(\frac{1\pm dn(\xi)}{sn(\xi)}\right)$.

Note that, when $m \to 0$ we obtain constant solutions, when $m \to 1$, we obtain $for\ \ \beta < 0$,

$$u_{82,83}(x,y,z,t) = -\frac{1}{2} + \frac{1}{2}\left(\frac{\tanh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\ \right)}{1\pm \operatorname{sech}\left(\frac{\sqrt{2\beta}}{6}\ \left(x+y+z+\frac{1}{6}\beta t\right)\ \right)}\right),$$

$$u_{84,85}(x,y,z,t) = -\frac{1}{2} - \frac{1}{2}\left(\frac{\tanh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\ \right)}{1\pm \operatorname{sech}\left(\frac{\sqrt{2\beta}}{6}\ \left(x+y+z+\frac{1}{6}\beta t\right)\ \right)}\right),\quad for\ \ \beta > 0\ ,$$

$$u_{86,87}(x,y,z,t) = -\ \frac{1}{2} +\ \frac{1}{2}\ \left(\frac{i\tan\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\ \right)}{1\pm \sec\left(\frac{\sqrt{2\beta}}{6}\ \left(x+y+z+\frac{1}{6}\beta t\right)\ \right)}\right),$$

$$u_{88,89}(x,y,z,t) = -\ \frac{1}{2} -\ \frac{1}{2}\ \left(\frac{i\tan\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\ \right)}{1\pm \sec\left(\frac{\sqrt{2\beta}}{6}\ \left(x+y+z+\frac{1}{6}\beta t\right)\ \right)}\right).$$

Using Eq. (10), the solution of Eq. (4.1) when $\left[p=\frac{m^2+1}{2},\ q=\frac{m^2-1}{2},\ r=\frac{1-m^2}{4}\right]$ and the sets of solutions (3) -(4), we get $u_{90,\dots 93}(x,y,z,t)=a_0+a_1\left(\frac{dn(\xi)}{1\pm m\,sn(\xi)}\right)+b_1\left(\frac{1\pm m\,sn(\xi)}{dn(\xi)}\right)$.

Note that, when $m\to 0$ we obtain constant solution, also when $m\to 1$, we obtain constant solutions.

Using Eq. (10), the solution of Eq. (4.1) when $\left[p=\frac{m^2+1}{2},\ q=\frac{-1}{2},\ r=\frac{-(1-m^2)^2}{4}\right]$ and the sets of solutions (3)-(4), we get $u_{94,\dots 97}(x,y,z,t)=a_0+a_1(m\,cn(\xi)\pm dn)+\frac{b_1}{(m\,cn(\xi)\pm dn)}$.

Note that, when $m\to 0$ we obtain constant solution, when $m\to 1$ we obtain $[u_{2,3}(x,y,z,t), and\ \ u_{4,5}(x,y,z,t)\ ]$

Using Eq. (10), the solution of Eq. (4.1) when $\left[p=\frac{m^2+1}{2},\ q=\frac{1-m^2}{2},\ r=\frac{1-m^2}{4}\right]$ and the sets of solutions (3) -(4), we get $u_{98,\dots,101}(x,y,z,t)=a_0+a_1\left(\frac{cn(\xi)}{1\pm sn(\xi)}\right)+b_1\left(\frac{1\pm sn(\xi)}{cn(\xi)}\right)$.

Note that, when $m\to 1$ we obtain constant solution, when $m\to 0$ we obtain $for\ \beta>0$,

$$u_{102,103}(x,y,z,t)=-\frac{1}{2}+\frac{i}{2}\left(\frac{\cos\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{1\pm\sin\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}\right),$$

$$u_{104,105}(x,y,z,t)=-\frac{1}{2}-\frac{i}{2}\left(\frac{\cos\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{1\pm\sin\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}\right),$$

$for\ \beta<0$,

$$u_{106,107}(x,y,z,t)=-\frac{1}{2}+\frac{i}{2}\left(\frac{\cosh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{1\pm i\sinh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}\right),$$

$$u_{108,109}(x,y,z,t)=-\frac{1}{2}-\frac{i}{2}\left(\frac{\cosh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{1\pm i\sinh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}\right).$$

Using Eq.(10), the solution of Eq. (4.1) when $\left[p=\frac{1-2m^2}{2},\ q=\frac{1}{2},\ r=\frac{m^2}{4}\right]$ and the sets of solutions (3) -(4), we get $u_{110,\dots,113}(x,y,z,t)=a_0+a_1(m\,sn(\xi)\pm i\,dn(\xi))+\frac{b_1}{(m\,sn(\xi)\pm i\,dn(\xi))}$.

Note that, when $m\to 0$ we obtain constant solutions, when $m\to 1$ we obtain $[u_{70,71}(x,y,z,t),\dots,u_{76,77}(x,y,z,t)]$.

Using Eq.(10), the solution of Eq. (4.1) when $\left[p=\frac{m^2-2}{2},\ q=\frac{m^2}{2},\ r=\frac{m^2}{4}\right]$ and the sets of solutions (3) -(4), we get

$$u_{114,\dots,117}(x,y,z,t)=a_0+a_1\big(m\,sn(\xi)\pm i\,cn(\xi)\big)+\frac{b_1}{(m\,sn(\xi)\pm i\,cn(\xi))}.$$

Note that, when $m\to 0$ we obtain constant solutions, when $m\to 1$, we obtain $[u_{70,71}(x,y,z,t),\dots,u_{76,77}(x,y,z,t)]$.

Using Eq. (10), the solution of Eq. (4.1) when $\left[p=\frac{m^2-2}{2},\ q=\frac{1}{2},\ r=\frac{m^4}{4}\right]$ and the sets of solutions (3)-(4), we get $u_{118,\dots,121}(x,y,z,t)=a_0+a_1(ns(\xi)\pm ds(\xi))+\frac{b_1}{(ns(\xi)\pm ds(\xi))}$).

Note that, when $m\to 0$, we obtain $[u_{22,23}(x,y,z,t)$ and $u_{24,25}(x,y,z,t)]$, when $m\to 1$ we obtain $for\ \beta>0$,

$$u_{122,123}(x,y,z,t)=-\frac{1}{2}+\frac{i}{2}\left(\cot\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm\csc\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\right),$$

$$u_{124,125}(x,y,z,t)=-\frac{1}{2}+\frac{i}{2}\left(\cot\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm\csc\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\right),$$

$for\ \beta<0$,

$$u_{126,127}(x,y,z,t)=-\frac{1}{2}+\frac{1}{2}\left(\coth\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm\operatorname{csch}\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\right),$$

$$u_{128,129}(x,y,z,t)=-\frac{1}{2}+\frac{1}{2}\left(\coth\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm\operatorname{csch}\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\right).$$

Using Eq.(10), the solution of Eq.(4.1) when $\left[p=\frac{1-2m^2}{2},\ q=\frac{1}{2},\ r=\frac{1}{4}\right]$ and the sets of solutions (3)-(4), we get $u_{130,\dots,133}(x,y,z,t)=a_0+a_1(ns(\xi)\pm cs(\xi))+\frac{b_1}{(ns(\xi)\pm cs(\xi))})$.

Note that, when $m\to 0$ we obtain, $[\,u_{122,123}(x,y,z,t)\,,\dots,u_{128,129}(x,y,z,t)]$,also when $m\to 1$ we obtain $[\,u_{126,127}(x,y,z,t)\,,\dots,u_{128,129}(x,y,z,t)]$.

Using Eq.(10), the solution of Eq. (4) when $\left[p=\frac{1-2m^2}{2},\ q=\frac{1}{2},\ r=\frac{1}{4}\right]$ and the sets of solutions (3)-(4), we get

$$u_{134,\dots,137}(x,y,z,t)=a_0+a_1\left(\frac{cn(\xi)}{\sqrt{1-m^2}\,sn(\xi)\pm dn(\xi)}\right)+b_1\left(\frac{\sqrt{1-m^2}\,sn(\xi)\pm dn(\xi)}{cn(\xi)}\right).$$

Note that, when $m\to 1$ we obtain constant solutions, when $m\to 0$, we obtain $[u_{102,103}(x,y,z,t),\dots,u_{108,109}(x,y,z,t)]$.

Using Eq.(10), the solution of Eq.(4.1) when $\left[p=\frac{1+m^2}{2},\ q=\frac{(1-m^2)^2}{2},\ r=\frac{1}{4}\right]$ and the sets of solutions (3) -(4), we get $u_{138,\dots,141}(x,y,z,t)=a_0+a_1\left(\frac{sn(\xi)}{cn(\xi)\pm dn(\xi)}\right)+b_1\left(\frac{cn(\xi)\pm dn(\xi)}{sn(\xi)}\right)$.

Note that, when $m\to 1$ we obtain constant solutions, when $m\to 0$, we obtain $\beta>0$,

$$u_{142,143}(x,y,z,t)=-\frac{1}{2}+\frac{i}{2}\left(\frac{\sin\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{\cos\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm 1}\right),$$

$$u_{144,145}(x,y,z,t)=-\frac{1}{2}-\frac{i}{2}\left(\frac{\sin\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{\cos\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm 1}\right),$$

$for\ \beta<0$,

$$u_{146,147}(x,y,z,t)=-\frac{1}{2}+\frac{1}{2}\left(\frac{\sinh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{\cosh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm 1}\right),$$

$$u_{148,149}(x,y,z,t)=-\frac{1}{2}-\frac{1}{2}\left(\frac{\sinh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{\cosh\left(\frac{\sqrt{2\beta}}{6}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm 1}\right).$$

Using Eq.(10), the solution of Eq. (4) when $\left[p=\frac{m^2-2}{2},\ q=\frac{m^2}{2},\ r=\frac{1}{4}\right]$ and the sets of solutions (3)-(4), we get $u_{150,\dots,153}(x,y,z,t)=a_0+a_1\left(\frac{cn(\xi)}{\sqrt{1-m^2}\pm dn(\xi)}\right)+b_1\left(\frac{\sqrt{1-m^2}\pm dn(\xi)}{cn(\xi)}\right)$.

Note that, when $m\to 1$ we obtain constant solutions, also when $m\to 0$, we obtain constant solutions.

Using Eq. (10), the solution of Eq. (4.1) when $\left[p=\frac{m^2-2}{2},\ q=\frac{m^2}{2},\ r=\frac{m^2}{4}\right]$ and the sets of solutions (3) -(4), we get

$$u_{154,\dots,157}(x,y,z,t)=a_0+a_1(\sqrt{m^2-1}\,sd(\xi)\pm cd(\xi))+\frac{b_1}{(\sqrt{m^2-1}\,sd(\xi)\pm cd(\xi))}.$$

Note that, when $m \to 1$ we obtain constant solutions, also when $m \to 0$, we obtain constant solutions.

Using Eq.(10), the solution of Eq. (4) when $\left[p = \frac{1-2m^2}{2},\ q = \frac{1}{2},\ r = \frac{1}{4}\right]$ and the sets of solutions (3)-(4), we get

$$u_{158,\dots,161}(x,y,z,t) = a_0 + a_1(m\, cd(\xi) \pm i\sqrt{m^2-1}\ nd(\xi)) + \frac{b_1}{(m\, cd(\xi) \pm i\sqrt{m^2-1}\ nd(\xi))}.$$

Note that, when $m \to 1$, we obtain constant solutions, also when $m \to 0$ we obtain constant solutions.

Using Eq.(10), the solution of Eq. (4) when $\left[p = \frac{1-2m^2}{2},\ q = \frac{1}{2},\ r = \frac{1}{4}\right]$ and the sets of solutions (3)-(4), we get

$$u_{162,\dots,165}(x,y,z,t) = a_0 + a_1(sc(\xi) \pm\ dc(\xi)) + \frac{b_1}{(sc(\xi) \pm dc(\xi))}.$$

Note that, when $m \to 1$, we obtain constant solutions, when $m \to 0$, we obtain $[u_{70,71}(x,y,z,t), \dots, u_{76,77}(x,y,z,t)]$.

Using Eq.(10), the solution of Eq. (4) when $\left[p = \frac{m^2+1}{2},\ q = \frac{m^2-1}{2},\ r = \frac{m^2-1}{4}\right]$ and the sets of solutions (3)-(4), we get

$$u_{166,\dots,169}(x,y,z,t) = a_0 + a_1(m\, sd(\xi) \pm\ nd(\xi)) + \frac{b_1}{(m\, sd(\xi) \pm nd(\xi))}.$$

Note that, when $m \to 1$ we obtain constant solutions, also when $m \to 0$ we obtain constant solutions.

Using Eq. (10), the solution of Eq. (4.1) when $\left[p = \frac{m^2+1}{2},\ q = \frac{(1-m^2)^2}{2},\ r = \frac{1}{4}\right]$ and the sets of solutions (3) -(4), we get

$$u_{170,\dots,173}(x,y,z,t) = a_0 + a_1\big(ds(\xi) \pm\ cs(\xi)\big) + \frac{b_1}{\big(ds(\xi) \pm cs(\xi)\big)}.$$

Note that, when $m \to 1$ we obtain constant solutions, when $m \to 0$ we obtain $[u_{22,23}\ (x,y,z,t), and\ \ u_{24,25}(x,y,z,t)]$.

Using Eq. (10), the solution of Eq. (4.1) when $\left[p = \frac{m^2-2}{2},\ q = \frac{1}{2},\ r = \frac{m^2}{4}\right]$ and the sets of solutions (3) -(4), we get

$$u_{174,\dots,177}(x,y,z,t) = a_0 + a_1(dc(\xi) \pm \sqrt{m^2-1}\, nc(\xi)) + \frac{b_1}{(dc(\xi) \pm \sqrt{m^2-1}\ nc(\xi))}.$$

Note that, when $m \to 1$ we obtain constant solutions, when $m \to 0$ we obtain $[u_{2,3}(x,y,z,t)\ and\ u_{4,5}(x,y,z,t)\ ]$.

Using Eq.(10), the solution of Eq.(4) when $[p = 2-4m^2,\ q = 2,\ r = 1]$ and the sets of solutions (3) -(4), we get $u_{178,..,181}((x,y,z,t) = a_0 + a_1\left(\frac{sn(\xi)dn(\xi)}{cn(\xi)}\right) + b_1\left(\frac{cn(\xi)}{sn(\xi)dn(\xi)}\right)$.

Note that, when $m \to 1$, we obtain $[u_{6,7}(x,y,z,t)$ and $u_{8,9}(x,y,z,t)]$,also when $m \to 0$ we obtain $\left[\ u_{6,7}(x,y,z,t)\ and\ u_{8,9}(x,y,z,t)\right]$.

Using Eq.(10), the solution of Eq. (4) when $[p = 2m^2, -4\ \ q = 2m^2,\ r = 1]$ and the sets of solutions (3)-(4), we get $u_{182,\dots,185}(x,y,z,t) = a_0 + a_1\left(\frac{sn(\xi)cn(\xi)}{dn(\xi)}\right) + b_1\left(\frac{sn(\xi)cn(\xi)}{dn(\xi)}\right)$.

Note that, when $m \to 1$, we obtain $[u_{6,7}(x,y,z,t)\ and\ u_{8,9}(x,y,z,t)]$,when $m \to 0$ we obtain constant solutions.

Using Eq. (10), the solution of Eq. (4.1) when $[p = 2m^2+2,\ q = 2,\ r = 1-2m^2+m^4]$ and the sets of solutions (3) -(4), we get

$$u_{186,\dots,189}(x,y,z,t)=a_0+a_1\left(\frac{dn(\xi)cn(\xi)}{sn(\xi)}\right)+b_1\left(\frac{dn(\xi)cn(\xi)}{sn(\xi)}\right).$$

Note that, when $m\to 0$ we obtain $[u_{10,11}(x,y,z,t)\ and\ u_{12,13}(x,y,z,t)]$, when $m\to 1$ we obtain $for\ \beta>0$,

$$u_{190,191}(x,y,z,t)=-\frac{1}{2}\pm\frac{\sqrt{2}\,i}{4}\ \frac{sech^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{tanh\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)},$$

$for\ \beta<0$,

$$u_{192,193}(x,y,z,t)=-\frac{1}{2}\pm\frac{\sqrt{2}}{4}\ \frac{sec^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{tan\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}\ .$$

Using Eq.(10), the solution of Eq.(4) when $\left[p=\frac{m^2+1}{2+3m^2},\ q=\frac{A^2(m^2-1)^2}{2},\ r=\frac{A^2(m^2-1)^2}{4A^2}\right]$ and the sets of solutions (3)-(4), we get

$$u_{194,\dots,197}(x,y,z,t)=a_0+a_1\left(\frac{dn(\xi)cn(\xi)}{A(1+sn(\xi))(1+m\,sn(\xi))}\right)+b_1\left(\frac{A(1+sn(\xi))(1+m\,sn(\xi))}{dn(\xi)cn(\xi)}\right).$$

Note that, when $m\to 0\ and\ A=1$ we obtain $[u_{102,103}(x,t),\dots,u_{108,109}(x,t)]$, when $m\to 1$ we obtain constant solutions.

Using Eq.(10), the solution of Eq.(4) when $\left[p=\frac{m^2+1}{2-3m^2},\ q=\frac{A^2(m^2+1)^2}{2},\ r=\frac{A^2(m^2+1)^2}{4A^2}\right]$ and the sets of solutions (3)-(4), we get

$$u_{198,\dots,201}(x,y,z,t)=a_0+a_1\left(\frac{dn(\xi)cn(\xi)}{A(1+sn(\xi))(1-m\,sn(\xi))}\right)+b_1\left(\frac{A(1+sn(\xi))(1-m\,sn(\xi))}{dn(\xi)cn(\xi)}\right).$$

Note that, when $m\to 0$ and $A=1$ we obtain $[u_{102,103}(x,y,z,t),\dots,u_{108,109}(x,y,z,t)]$, when $m\to 1\ and\ A=1$, we obtain $for\ \ \beta<0$,

$$u_{202,203}(x,y,z,t)=-\frac{1}{2}\pm\frac{1}{2}\left(\frac{sech^2\left(\frac{\sqrt{2\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{1-tanh^2\left(\frac{\sqrt{2\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}\right),$$

$for\ \ \beta>0$,

$$u_{204,205}(x,y,z,t)=-\frac{1}{2}\pm\frac{1}{2}\left(\frac{sec^2\left(\frac{\sqrt{2\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{1-i\,tan^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}\right).$$

Using Eq.(10), the solution of Eq.(4) when $\left[p=6m-m^2-1, q=\frac{-8}{m},\ r=-2m^3+m^4+m^2\right]$ and the sets of solutions (3)-(4), we get

$$u_{206,\dots,209}(x,y,z,t)=a_0+a_1\left(\frac{m\,cn(\xi)\,dn(\xi)}{m\,sn^2(\xi)+1}\right)+b_1\left(\frac{m\,sn^2(\xi)+1}{m\,cn(\xi)\,dn(\xi)}\right).$$

Note that, when $m\to 0$ we obtain constant solutions, when $m\to 1$, we obtain $for\ \ \beta>0$,

$$u_{210,211}(x,y,z,t)=-\frac{1}{2}+\frac{\sqrt{2}}{2}\left(\frac{sech^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{tanh^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm 1}\right),$$

$$u_{212,213}(x,y,z,t)=-\frac{1}{2}-\frac{\sqrt{2}}{2}\left(\frac{sech^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{tanh^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm 1}\right),$$

$for\ \ \beta<0$,

$$u_{214,215}(x,y,z,t)=-\frac{1}{2}+\frac{\sqrt{2}}{2}\left(\frac{sec^2\left(\frac{\sqrt{2\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{i\,tan^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm 1}\right),$$

$$u_{216,217}(x,y,z,t)=-\frac{1}{2}-\frac{\sqrt{2}}{2}\left(\frac{sec^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{i\,tan^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)\pm 1}\right).$$

Using Eq.(10), the solution of Eq.(4) when $\left[p = -6m - m^2 - 1, q = \frac{8}{m},\ r = 2m^3 + m^4 + m^2\right]$ and the sets of solutions (3) -(4), we get

$$u_{218,\dots,221}(x,y,z,t) = a_0 + a_1\left(\frac{m\, cn(\xi)\, dn(\xi)}{m\, sn^2(\xi)-1}\right) + b_1\left(\frac{m\, sn^2(\xi)-1}{m\, cn(\xi)\, dn(\xi)}\right).$$

Note that, when $m \to 0$ we obtain constant solutions, when $m \to 1$, we obtain $for\ \ \beta < 0$,

$$u_{222,223}(x,y,z,t) = -\frac{1}{2} \pm \frac{1}{2}\left(\frac{sech^2\left(\frac{\sqrt{2\beta}}{24}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{tanh^2\left(\frac{\sqrt{2\beta}}{24}\left(x+y+z+\frac{1}{6}\beta t\right)\right)-1}\right),$$

$for\ \ \beta > 0$,

$$u_{224,225}(x,y,z,t) = -\frac{1}{2} \pm \frac{1}{2}\left(\frac{sec^2\left(\frac{\sqrt{2\beta}}{24}\left(x+y+z+\frac{1}{6}\beta t\right)\right)}{i\, tan^2\left(\frac{\sqrt{2\beta}}{24}\left(x+y+z+\frac{1}{6}\beta t\right)\right)-1}\right).$$

Using Eq.(10), the solution of Eq.(4) when $\left[p = 2m^2 + 2,\ q = -2(m^2 + 2m + 1)B^2,\ r = \frac{2m-m^2-1}{B^2}\right]$ and the sets of solutions (3) -(4), we get

$$u_{226,\dots,229}(x,y,z,t) = a_0 + a_1\left(\frac{m\, sn^2(\xi)-1}{B(m\, sn^2(\xi)+1)}\right) + b_1\left(\frac{B(m\, sn^2(\xi)+1)}{m\, sn^2(\xi)-1}\right).$$

Note that, when $m \to 0$ we obtain constant solutions, when $m \to 1$ we obtain $for\ \ \beta > 0$,

$$u_{230,231}(x,y,z,t) = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}\left(\frac{tanh^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)-1}{tanh^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)+1}\right),$$

$for\ \ \beta < 0$,

$$u_{232,233}(x,y,z,t) = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}\left(\frac{i\, tan^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)-1}{i\, tan^2\left(\frac{\sqrt{\beta}}{12}\left(x+y+z+\frac{1}{6}\beta t\right)\right)+1}\right).$$

Using Eq.(10), the solution of Eq.(4) when $\left[p = 2m^2 + 2,\ q = -2(m^2 - 2m + 1)B^2,\ r = \frac{-(2m+m^2+1)}{B^2}\right]$ and the sets of solutions (3)-(4), we get

$$u_{234,\dots,237}(x,y,z,t) = a_0 + a_1\left(\frac{m\, sn^2(\xi)+1}{B(m\, sn^2(\xi)-1)}\right) + b_1\left(\frac{B(m\, sn^2(\xi)-1)}{m\, sn^2(\xi)+1}\right).$$

Note that, when $m \to 0$ we obtain constant solutions, also when $m \to 1$, we obtain constant solutions.

Using Eq.(10), the solution of Eq.(4) when $\left[r = \frac{2m^2p^2}{(m^2+1)^2 q},\ p < 0\, , q > 0\right]$ and the sets of solutions (3)-(4), we get

$$u_{238,\dots,241}(x,y,z,t) = a_0 + a_1\left(\sqrt{\frac{-2m^2 p}{(m^2+1)q}}\, sn\left(\sqrt{\frac{-p}{(m^2+1)}}\,\xi\right)\right) + b_1\left(\sqrt{\frac{-2m^2 p}{(m^2+1)q}}\, sn\left(\sqrt{\frac{-p}{(m^2+1)}}\,\xi\right)\right)^{-1}.$$

Note that, when $m \to 0$ we obtain constant solutions, when $m \to 1$, we obtain $[\, u_{6,7}(x,y,z,t)\ and\ u_{8,9}(x,y,z,t)\,]$.

Using Eq.(10), the solution of Eq.(4) when $\left[r = \frac{2p^2(1-m^2)}{(m^2-2)^{\,2}\, q},\ p > 0\, , q < 0\right]$ and the sets of solutions (3)-(4), we get

$$u_{242,\dots,245}(x,y,z,t) = a_0 + a_1\left(\sqrt{\frac{-2p}{(2-m^2)q}}\, dn\left(\sqrt{\frac{p}{(2-m^2)}}\,\xi\right)\right) + b_1\left(\sqrt{\frac{-2p}{(2-m^2)q}}\, dn\left(\sqrt{\frac{p}{(2-m^2)}}\,\xi\right)\right)^{-1}.$$

Note that, when $m \to 0$ we obtain constant solutions, when $m \to 1$, we obtain $[\, u_{2,3}(x,y,z,t)\ and\ u_{4,5}(x,y,z,t)]$.

Using Eq.(10), the solution of Eq.(4) when $\left[r = \frac{2m^2p^2(m^2-1)}{(2m^2-1)^2q},\ p > 0\,, q < 0\right]$ and the sets of solutions (3)-(4) , we get

$$u_{246,\dots,249}(x,y,z,t) = a_0 + a_1\left(\sqrt{\frac{-2m^2p}{(2m^2-1)q}}\,cn\left(\sqrt{\frac{p}{(2m^2-1)}}\,\xi\right)\right) + b_1\left(\sqrt{\frac{-2p}{(2m^2-1)q}}\,cn\left(\sqrt{\frac{p}{(2m^2-1)}}\,\xi\right)\right)^{-1}.$$

Note that, when $m \to 0$ we obtain constant solutions, when $m \to 1$ we obtain[$u_{2,3}(x,y,z,t)$ *and* $u_{4,5}(x,y,z,t)$].

4. Conclusion:

In this paper, the mapping method has been successfully implemented to find new traveling wave solutions for our new proposed equation namely, a combined of (KDV-ZK and mKDV-ZK) equation. The results show that this method is a powerful Mathematical tool for obtaining exact solutions for our equation. It is also a promising method to solve other nonlinear partial differential equations.

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حلول دقيقة جديدة لمعادلة كورتيويغ ـ دي فاريس- زاخاروف- كوزنيتسوف (KdV-ZK) المقترنة ومعادلة كورتيويغ- دي فاريس- زاخاروف- كوزنيتسوف (mKdv-ZK) المعدلة

محمد سالم العمري[1]
قسم الرياضيات، كلية التربية عدن، جامعة عدن

أحلام سعد محمد[2]
قسم الرياضيات، كلية التربية عدن، جامعة عدن

الملخص: في هذه الورقة، نقدم نموذجًا لمعادلة كورتيويغ - دي فاريس - زاخاروف - كوزنيتسوف (KdV-ZK) المركبة ومعادلة كورتيويغ - دي فاريس - زاخاروف - كوزنيتسوف (mKdv-ZK) المعدلة، ونطبق طريقة التعيين لحل المعادلة. يتم الحصول على حلول دقيقة للموجات المتنقلة ويتم التعبير عنها من حيث الدوال الزائدية والدوال المثلثية والدوال الكسرية.

الكلمات المفتاحية: معادلة (Kdv–ZK)، معادلة (mKdv–ZK)، معادلة (mKdv–ZK و c(kdv- Zk ، الحل الدقيق وطريقة رسم الخرائط.