Generalized R^h-Recurrent Spaces in Finsler Geometry: Properties, Identities, and Characterizations

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Abstract: In this paper, we investigate a class of Finsler spaces, termed generalized R^h-recurrent spaces (denoted as GR^h-RF_n), where the Cartan's third curvature tensor satisfies a specific condition involving non-null covariant vector fields. We provide several characterizations and properties of these spaces, starting with the equivalence of two distinct forms of the curvature condition, and prove key theorems about the non-vanishing of essential tensors like the Ricci tensor, curvature vector, deviation tensor, and the curvature scalar. Furthermore, we derive important identities associated with the covariant differentiation of tensors in these spaces, such as the h-covariant derivative of the h(v)-torsion tensor and deviation tensor. These identities play a crucial role in understanding the geometric structure of GR^h-RF_n, and the results contribute to the broader theory of recurrent spaces in Finsler geometry. The paper concludes by presenting several new identities, which hold in GR^h-RF_n, thus advancing the theoretical framework of generalized recurrent spaces in differential geometry.

Keywords: Finsler space, Generalized R^h-recurrent space**,** h(v)-torsion tensor.

1. Introduction

In the field of Finsler geometry, recurrent spaces have been a topic of significant interest due to their deep structural and geometric properties. This paper focuses on a particular class of Finsler spaces, referred to as generalized R^h-recurrent spaces (denoted as GR^h-RF_n). These spaces are characterized by a specific condition on the Cartan's third curvature tensor, where the non-null covariant vector fields λ_l , μ_l and γ_l play a crucial role. By exploring various relationships and conditions, such as transvection by the metric tensor and covariant derivatives, we derive essential results regarding the properties of GR^h-RF_n, including the non-vanishing of important geometric tensors like the Ricci tensor, curvature vector, deviation tensor, and the curvature scalar. The study not only provides a clearer understanding of the structure of these spaces but also contributes new identities that hold within GR^h-RF_n, which further enhance the broader theory of recurrent spaces in Finsler geometry.

In recent years, the study of Finsler spaces, particularly in the context of recurrent and projective curvature tensors, has gained significant attention in the field of differential geometry and mathematical physics. Researchers such as AL-Qashbari et al. (2024, 2025) have made considerable contributions to the understanding of R-projective curvature tensors and their relations in recurrent Finsler spaces, with notable works focusing on mixed birecurrent Finsler spaces and generalized recurrent tensor fields. These studies explore the Lie derivative's role in various curvature tensors, as seen in works by AL-Qashbari and Baleedi (2023) on the M-projective curvature tensor and K-curvature inheritance. Moreover, Atashafrouz and Najafi (2021) delved into D-recurrent Finsler metrics, offering a framework that further expands the general theory of curvature tensors in Finsler geometry. The contributions by Ghadle et al. (2024) and Opondo (2021) provide additional insights into generalized BP-recurrent spaces, laying the groundwork for deeper analyses of tensor fields and their applications. This body of work, complemented by foundational texts such as Rund's (1959) The Differential Geometry of Finsler Spaces, has paved the way for this research, which seeks to extend these ideas and explore new relationships in higher-order generalized Finsler spaces. In particular, this paper will focus on the study of M-projective curvature tensors in the context of $\mathrm{GBK\text{-}5RF_n}$ spaces, using Lie derivatives as a critical tool in understanding their geometric properties.

Let us consider an n-dimensional Finsler space equipped with the metric function F satisfying the requisite conditions [15]. Let consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ^{*i}_{jk} and Berwald's connection parameters G^i_{jk} . These are symmetric in their lower indices and positively homogeneous of degree zero in the directional arguments.

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by

(1.1) $$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 \;, & if \quad i = k \;, \\ 0 \;, & if \quad i \neq k \;. \end{cases}$$

The vectors y_i and y^i satisfies the following relations

(1.2) a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $g_{ij} = \dot{\partial}_i\, y_j = \dot{\partial}_j\, y_i$,

d) $g_{ij}\, y^j = \frac{1}{2}\,\dot{\partial}_i\, F^2 = F \dot{\partial}_i\, F$ and e) $\dot{\partial}_j\, y^i = \delta^i_j$.

The tensor C_{ijk} defined by

(1.3) $$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\,\dot{\partial}_i\,\dot{\partial}_j\,\dot{\partial}_k\, F^2$$

is known as (h) hv-torsion tensor. It is positively homogeneous of degree -1 in the directional arguments and symmetric in all its indices.

The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor C_{ijk} are related by

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ and b) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$.

The (v)hv-torsion tensor C^i_{ik} is also positively homogeneous of degree -1 in the directional arguments and symmetric in its lower indices.

É. Cartan deduced the h-covariant derivative for an arbitrary vector filed X^i with respect to x^k

(1.5) $\quad X^i_{|k} = \partial_k X^i - (\dot{\partial}_r X^i)\, G^r_k + X^r\, \Gamma^{*i}_{rk}$.

The metric tensor g_{ij} and the vector y^i are covariant constant with respect to a above process.

(1.6) $\quad$ a) $g_{ij|k} = 0$, b) $y^i_{|k} = 0$ and c) $g^{ij}_{|k} = 0$.

The process of h-covariant differentiation with respect to x^k commute with partial differentiation with respect to y^j for arbitrary vector filed X^i, according to

(1.7) $\quad \dot{\partial}_j \left(X^i_{|k}\right) - \left(\dot{\partial}_j X^i\right)_{|k} = X^r \left(\dot{\partial}_j\, \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk}$, where

(1.8) $\quad$ a) $\dot{\partial}_j \Gamma^{*r}_{hk} = \Gamma^{*r}_{jhk}$, b) $P^i_{kh}\, y^k = 0 = P^i_{kh}\, y^h$ and c) $P^i_{jkh}\, y^j = P^i_{jkh}$.

The tensor P^i_{kh} is called v(hv) –torsion tensor and its associate tensor P_{jkh} is given by

(1.9) $\quad g_{rj}\, P^r_{kh} = P_{kjh}$.

The quantities H^i_{jkh} and H^i_{kh} form the components of tensors and they called h-curvature tensor of Berwald (Berwald curvature tensor) and torsion tensor, respectively, and defined as follow:

(1.10) $\quad$ a) $H^i_{jkh} = \partial_j\, G^i_{kh} + G^r_{kh}\, G^i_{rj} + G^i_{rhj}\, G^r_k - h/k$ *

and $\quad$ b) $H^i_{kh} = \partial_h\, G^i_k + G^r_k\, C^i_{rh} - h/k$.

They are skew-symmetric in their lower indices, i.e. k and h . Also they are positively homogeneous of degree zero and one, respectively in their directional arguments. They are also related by

(1.11) $\quad$ a) $H^i_{jkh}\, y^j = H^i_{kh}$, b) $H^i_{jkh} = \dot{\partial}_j H^i_{kh}$ and c) $H^i_{jk} = \dot{\partial}_j\, H^i_k$.

These tensors were constructed initially by mean of the tensor H^i_h , called the deviation tensor, given by

(1.12) $\quad H^i_h = 2\, \partial_h\, G^i - \partial_r\, G^i_h\, y^r + 2\, G^i_{hs}\, G^s - G^i_s\, G^s_h$.

The deviation tensor H^i_h is positively homogeneous of degree two in the directional arguments.

In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (1.11) and (1.12), we have the following:

(1.13) $\quad$ a) $H^i_{jk}\, y^j = -\, H^i_{kj}\, y^j = H^i_k$ and b) $y_i\, H^i_j = 0$.

The quantities H^i_{jkh} and H^i_{kh} are satisfies the following

(1.14) $\quad$ a) $H_{ijkh} = g_{jr}\, H^r_{ihk}$, b) $H_{jk.h} = g_{jr} H^r_{hk}$ and c) $y_i\, H^i_{jk} = 0$.

Cartan's third curvature tensor R^i_{jkh} satisfies the identity known as Bianchi identity

* $-h/k$ means the subtraction from the former term by interchanging the indices h and k .

(1.15) a) $R^i_{jkh|s} + R^i_{jsk|h} + R^i_{jhs|k} + \left(R^r_{mhs} P^i_{jkr} + R^r_{mkh} P^i_{jsr} + R^r_{msk} P^i_{jhr}\right) y^m = 0$

and b) $R_{ijhk} + R_{ihkj} + R_{ikjh} + \left(C_{ijs} K^s_{rhk} + C_{ihs} K^s_{rkj} + C_{iks} K^s_{rjh}\right) y^r = 0$,

where c) $P^r_{ijs} = \dot{\partial}_s \Gamma^{*r}_{ij} - C^r_{is|j} + C^r_{im} C^m_{js|k} y^k$.

The Ricci tensor R_{jk} , the deviation tensor R^r_h and the curvature scalar R of the curvature tensor R^i_{jkh} are given by

(1.16) a) $R^i_{jkh} y^j = H^i_{hk} = K^i_{jhk} y^j$, b) $R_{ijhk} = g_{rj} R^r_{ihk}$, c) $R_{jkhm} y^j = H_{kh.m}$,

d) $R^r_{ihk} = g^{jr} R_{ijhk}$, e) $R^i_{jkh} g^{jk} = R^i_h$ and f) $g_{ip} R^i_j = R_{jp}$.

The contracted tensor R_{kh}(Ricci tensor) and R_k (Curvature vector) are also connected by

(1.17) a) $R_{jk} y^k = R_j$, b) $R_{jk} y^j = H_k$ and c) $R^i_{jki} = R_{jk}$.

Also this tensor satisfies the following relation too

(1.18) a) $R^i_{jkh} = K^i_{jkh} + C^i_{js} K^s_{rhk} y^r$, b) $R_{ijkh} = K_{ijkh} + C_{ijs} H^s_{kh}$

and c) $R_{jkhm} y^j = H_{kh.m}$, where

R_{ijkh} is the associate curvature tensor of R^i_{jkh} . Cartan's fourth curvature tensor K^i_{jkh} and its associate curvature tensor of K_{ijkh}satisfy the following known as Bianchi identities

(1.19) a) $K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0$ and b) $K_{jrkh} + K_{hrjk} + K_{krhj} = 0$.

2. On Generalized R^h-Recurrent Space

Let us consider a Finsler space F_n whose Cartan's third curvature tensor R^i_{jkh} satisfies the following condition

(2.1) $R^i_{jkh|l} = \lambda_l R^i_{jkh} + \mu_l\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \frac{1}{4}\gamma_l\left(R^i_h g_{jk} - R^i_k g_{jh}\right)$, $R^i_{jkh} \neq 0$, where

λ_l , μ_l and γ_l are non-null covariant vectors field. We shall call such space as a generalized R^h-recurrent space. We shall denote it briefly by GR^h- RF_n .

Transvecting of (2.1) by the metric tensor g_{ip} , using (1.6a), (1.16b), (1.16f) and (1.1), we get

(2.2) $R_{jpkh|l} = \lambda_l R_{jpkh} + \mu_l\left(g_{hp}\, g_{jk} - g_{kp}\, g_{jh}\right) + \frac{1}{4}\gamma_l\left(R_{hp} g_{jk} - R_{kp} g_{jh}\right)$.

Conversely, the transvection of the condition (2.2) by the associate tensor g^{ip} of the metric tensor g_{ip} , yields the condition (2.1). Thus, the condition (2.2) is equivalent to the condition (2.1). Therefore, a generalized R^h- recurrent space characterized by the condition (2.2).

This compelling evidence leads us to conclude that

Theorem 2.1. An GR^h-RF_n space , may characterized by the condition (2.2).

Let us consider $G\,R^h$- $R\,F_n$ characterized by the condition (2.2).

Transvecting the condition (2.1) by y^j, using (1.6b), (1.16a) and (1.2a), we get

(2.3) $H^i_{kh|l} = \lambda_l\, H^i_{kh} + \mu_l\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) + \frac{1}{4}\gamma_l\left(R^i_h\, y_k - R^i_k\, y_h\right)$.

Further, transvecting (2.3) by y^k, using (1.6b), (1.13a), (1.2b) and in view of (1.1), we get

(2.4) $H^i_{h|l} = \lambda_l H^i_h + \mu_l(\delta^i_h F^2 - y_h y^i) + \frac{1}{4}\gamma_l(R^i_h F^2 - R^i_k y_h y^k)$.

Hence, it follows that the experiment was successful

Theorem 2.2. In GR^h-RF_n space, the h-covariant derivative of the h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h is given by the conditions (2.3) and (2.4), respectively.

Contracting the indices *i* and *h* in the condition (2.1), using (1.17c) and (1.1), we get

(2.5) $R_{jk|l} = \lambda_l R_{jk} + (n-1)\,\mu_l\, g_{jk} + \frac{1}{4}\gamma_l(R\, g_{jk} - R_{jk})$, where $R^i_i = R$.

Transvecting (2.5) by y^k , using (1.6b), (1.17a), (1.2b) and (1.2a), we get

(2.6) $R_{j|l} = \lambda_l R_j + (n-1)\,\mu_l\, y_j + \frac{1}{4}\gamma_l(R\, y_j - R_j)$.

Further, transvecting the condition (2.1) by the associate tensor g^{jk} of the metric tensor g_{jk} , using (1.6c), (1.16e) and in view of (1.1), we get

(2.7) $R^i_{h|l} = \lambda_l R^i_h + (n-1)\,\mu_l\,\delta^i_h + \frac{1}{4}\gamma_l(R^i_h\, g - R^i_k\,\delta^k_h)$.

Contracting the indices i and h in condition (2.7) and using (1.1), we get

(2.8) $R_{|l} = \lambda_l R + n\,(n-1)\,\mu_l + \frac{1}{4}\gamma_l(g-1)R$, where $R^r_r = R$.

The conditions (2.5), (2.6), (2.7) and (2.8), show that the Ricci tensor R_{jk} , the curvature vector R_j , the deviation tensor R^i_h and the curvature scalar R of a generalized R^h-recurrent space cannot vanish, because the vanishing of them imply the vanishing of the covariant vector field μ_l , i.e. $\mu_l = 0$, a contradiction.

This compelling evidence leads us to conclude that

Theorem 2.3. In $G\,R^h$-$R\,F_n$ space, the Ricci tensor R_{jk} , the curvature vector R_j , the deviation tensor R^i_h and the curvature scalar R are non-vanishing.

3. Certain Identities

In this section we shall obtain some identities in GR^h-RF_n .

Taking h-covariant differentiation of the formula (1.15b) with respect to x^l in the sense of Cartan and transvecting (1.15b) by the associate tensor g^{jp} of the metric tensor g_{jp} , using (1.6c), (1.16a), (1.16d) and (1.4a), we get

$$R^p_{ihk|l} + g^{jp}R_{ihkj|l} + g^{jp}R_{ikjh|l} + \left(C^p_{is}H^s_{hk} + g^{jp}C_{ihs}\,H^s_{kj} + g^{jp}C_{iks}\,H^s_{jh}\right)_{|l} = 0 \ .$$

Using equation (2.1) in the above equation, we get

(3.1) $\lambda_l R^p_{ihk} + \mu_l(\delta^p_k\, g_{ih} - \delta^i_h\, g_{ik}) + \frac{1}{4}\gamma_l(R^p_k g_{ih} - R^p_h g_{ik}) + g^{jp}\,R_{ihkj|l} + g^{jp}\,R_{ikjh|l}$
$+ \left(C^p_{is}\,H^s_{hk} + g^{jp}C_{ihs}\,H^s_{kj} + g^{jp}C_{iks}\,H^s_{jh}\right)_{|l} = 0$.

Transvecting (3.1) by y^i, using (1.6b), (1.16a), (1.18c), (1.4b) and (1.4a), we get

(3.2) $\lambda_l H^p_{hk} + \mu_l(\delta^p_k\, y_h - \delta^i_h\, y_k) + \frac{1}{4}\gamma_l(R^p_k y_h - R^p_h y_k) + g^{jp}H_{hk.j|l} + g^{jp}H_{kj.h|l} = 0$.

This compelling evidence leads us to conclude that

Theorem 3.1. In GR^h-RF_n space, the identities (3.1) and (3.2) hold good.

Using (1.16a) and (1.16b) in the identity (1.15b), we get

(3.3) $$g_{rj}R^r_{ihk} + g_{rh}R^r_{ikj} + g_{rk}R^r_{ijh} + C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} = 0 \ .$$

Now, transvecting (3.3) by y^r, using (1.2a) and (1.4b), we get

(3.4) $$y_j\, H^r_{hk} + y_h\, H^r_{kj} + y_k\, H^r_{jh} = 0 \ .$$

Also, transvecting (3.3) by y^i, using (1.16a), (1.14b) and (1.4b), we get

(3.5) $$H_{hj.k} + H_{kh.j} + H_{jk.h} = 0 \ .$$

Transvecting (3.4) by y^h , using (1.13a) and (1.2a), we get

(3.6) $$g_{rj}\, H^r_k - g_{rk}\, H^r_j = 0 \ .$$

Hence, it follows that the experiment was successful

Theorem 3.2. In GR^h-RF_n space, the identities (3.4), (3.5) and (3.6) hold good.

Using (1.16a) in the identity (1.15a), we get

(3.7) $$R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + H^s_{kh}\, P^r_{ijs} + H^s_{jk}\, P^r_{ihs} + H^s_{hj}\, P^r_{iks} = 0 \ .$$

In view of the condition (2.1), the identity (3.7), may be written as

(3.8) $$\lambda_h\, R^r_{ijk} + \lambda_k\, R^r_{ihj} + \lambda_j\, R^r_{ikh} + \mu_h\big(\delta^r_k g_{ij} - \delta^r_j\, g_{ik}\big) + \tfrac{1}{4}\gamma_h\big(R^r_k g_{ij} - R^r_j\, g_{ik}\big)$$
$$+ \mu_k\big(\delta^r_j\, g_{ih} - \delta^r_h g_{ij}\big) + \tfrac{1}{4}\gamma_k\big(R^r_j\, g_{ih} - R^r_h g_{ij}\big) + \mu_j(\delta^r_h g_{ik} - \delta^r_k g_{ih})$$
$$+ \tfrac{1}{4}\gamma_j(R^r_h g_{ik} - R^r_k g_{ih}) + \big(H^s_{kh}\, P^r_{ijs} + H^s_{jk}\, P^r_{ihs} + H^s_{hj} P^r_{iks}\big) = 0 \ .$$

Transvecting (3.8) by y^i , using (1.16a), (1.2a) and (1.8c), we get

(3.9) $$\lambda_h H^r_{jk} + \lambda_k\, H^r_{hj} + \lambda_j H^r_{kh} + \mu_h\big(\delta^r_k\, y_j - \delta^r_j\, y_k\big) + \tfrac{1}{4}\gamma_h\big(R^r_k\, y_j - R^r_j\, y_k\big)$$
$$+ \mu_k\big(\delta^r_j\, y_h - \delta^r_h\, y_j\big) + \tfrac{1}{4}\gamma_k\big(R^r_j\, y_h - R^r_h\, y_j\big) + \mu_j(\delta^r_h\, y_k - \delta^r_k\, y_h)$$
$$+ \tfrac{1}{4}\gamma_j(R^r_h\, y_k - R^r_k\, y_h) + \big(H^s_{kh}\, P^r_{js} + H^s_{jk}\, P^r_{hs} + H^s_{hj}\, P^r_{ks}\big) = 0 \ .$$

Transvecting (3.9) by y^j, using (1.13a), (1.2b), (1.1) and (1.8b), we get

(3.10) $$\lambda_h H^r_k - \lambda_k H^r_h + \lambda\, H^r_{kh} + \mu_h(\, \delta^r_k\, F^2 - y_k\, y^r) + \tfrac{1}{4}\gamma_h\big(R^r_k\, F^2 - R^r_j\, y_k y^j\big)$$
$$+ \mu_k(\, y_h\, y^r - \delta^r_h\, F^2) + \tfrac{1}{4}\gamma_k\big(R^r_j\, y_h\, y^j - R^r_h\, F^2\big) + \mu\, (\delta^r_h\, y_k - \delta^r_k\, y_h)$$
$$+ \tfrac{1}{4}\gamma\, (R^r_h\, y_k - R^r_k\, y_h) + (H^s_k\, P^r_{hs} - H^s_h\, P^r_{ks}) = 0 \ ,$$

where $\lambda_j\, y^j = \lambda$, $\gamma_j\, y^j = \gamma$ and $\mu_j\, y^j = \mu$.

Hence, it follows that the experiment was successful

Theorem 3.3. In GR^h-RF_n space, the identities (3.8), (3.9) and (3.10) hold good.

Transvecting (3.9) and (3.10) by the vector y_r , using (1.14c), (1.1), (1.13b) and (1.2b), we get

(3.11) $$\mu_h\big(\delta^r_k\, y_j - \delta^r_j\, y_k\big)\, y_r + \tfrac{1}{4}\gamma_h\big(R^r_k\, y_j - R^r_j\, y_k\big)\, y_r + \mu_k\big(\delta^r_j\, y_h - \delta^r_h\, y_j\big)\, y_r$$

$$+\tfrac{1}{4}\gamma_k\big(R^r_j\, y_h - R^r_h\, y_j\big)\, y_r + \mu_j(\delta^r_h\, y_k - \delta^r_k\, y_h)\, y_r + \tfrac{1}{4}\gamma_j(R^r_h\, y_k - R^r_k\, y_h)\, y_r$$

$$+\big(H^s_{kh}\, P^r_{js} + H^s_{jk}\, P^r_{hs} + H^s_{hj}\, P^r_{ks}\big)\, y_r = 0 \quad , \quad \text{and}$$

(3.12) $$\mu_h(y_r\delta^r_k\, F^2 - y_k\, F^2) + \tfrac{1}{4}\gamma_h\big(R^r_k\, F^2 - R^r_j\, y_k y^j\big)\, y_r + \mu_k(y_h\, F^2 - y_r\, \delta^r_h\, F^2)$$

$$+\tfrac{1}{4}\gamma_k\big(R^r_j\, g_{ih}\, y^j - R^r_h\, y_j\big)\, y_r + \mu\,(\delta^r_h\, y_k - \delta^r_k\, y_h)\, y_r$$

$$+\tfrac{1}{4}\gamma\,(R^r_h\, y_k - R^r_k\, y_h)\, y_r + H^s_k\, F^2 P^r_{hs} - H^s_h\, F^2 P^r_{ks} = 0 \quad \text{, respectively.}$$

Hence, it follows that the experiment was successful

Theorem 3.4. In GR^h-RF_n space, the identity (3.11) holds good.

Theorem 3.5. In GR^h-RF_n space , we have the identity (3.12).

Transvecting (3.8), (3.9) and (3.10) by the metric tensor g_{rm} , using (1.16b), (1.1), (1.9b), (1.14b), (1.9a) and (1.2a), we get

(3.13) $$\lambda_h R_{imjk} + \lambda_k R_{imhj} + \lambda_j R_{imkh} + \mu_h\big(g_{km}\, g_{ij} - g_{jm}\, g_{ik}\big) + \tfrac{1}{4}\gamma_h\big(R_{km}g_{ij} - R_{jm}g_{ik}\big)$$

$$+\mu_k\big(g_{jm}\, g_{ih} - g_{hm}\, g_{ij}\big) + \tfrac{1}{4}\gamma_k\big(R_{jm}g_{ih} - R_{hm}g_{ij}\big) + \mu_j(g_{hm}\, g_{ik} - g_{km}\, g_{ih})$$

$$+\tfrac{1}{4}\gamma_j(R_{hm}g_{ik} - R_{km}g_{ih}) + \big(H^s_{kh}\, P_{imjs} + H^s_{jk}\, P_{imhs} + H^s_{hj}\, P_{imks}\big) = 0 \quad ,$$

(3.14) $$\lambda_h H_{jm.k} + \lambda_k H_{hm.j} + \lambda_j H_{km.h} + \mu_h\big(\delta^r_k\, y_j - \delta^r_j\, y_k\big)\, g_{rm} + \tfrac{1}{4}\gamma_h\big(R_{km}y_j - R_{jm}y_k\big)$$

$$+\mu_k\big(\delta^r_j\, y_h - \delta^r_h\, y_j\big)\, g_{rm} + \tfrac{1}{4}\gamma_k\big(R_{jm}y_h - R_{hm}y_j\big) + \mu_j(\delta^r_h\, y_k - \delta^r_k\, y_h)\, g_{rm}$$

$$+\tfrac{1}{4}\gamma_j(R_{hm}y_k - R_{km}y_h) + \big(H^s_{kh}\, P_{jms} + H^s_{jk}\, P_{hms} + H^s_{hj}\, P_{kms}\big) = 0 \quad , \quad \text{and}$$

(3.15) $$g_{rm}(\lambda_h H^r_k - \lambda_k H^r_h + \lambda\, H^r_{kh}) + \mu_h(g_{km}F^2 - y_k\, y_m) + \tfrac{1}{4}\gamma_h(R_{km}F^2 - R_m y_k)$$

$$+\mu_k(\, y_h\, y_m - g_{hm}\, F^2) + \tfrac{1}{4}\gamma_k(R_m y_h - R_{hm}F^2) + \mu\,(g_{hm}\, y_k - g_{km}\, y_h)$$

$$+\tfrac{1}{4}\gamma\,(R_{hm}y_k - R_{km}y_h) + (H^s_k\, P_{hms} - H^s_h\, P_{kms}) = 0 \quad \text{, respectively.}$$

Hence, it follows that the experiment was successful

Theorem 3.6. In GR^h-RF_n space , the identities (3.13), (3.14) and (3.15) hold good.

4. Recommendations:

Exploration of Related Structures: It is recommended that future research extends the concept of GR^h-RF_n to explore its interactions with other geometric structures such as Riemannian spaces or generalized Berwald spaces. This could potentially lead to new insights or deeper connections between different branches of differential geometry.

Generalizations: The work could be expanded by generalizing the conditions on the Cartan curvature tensor to include other forms of covariant vector fields or higher-order derivatives. This would allow for the exploration of more complex recurrent spaces and their potential applications.

Further Characterization of Tensors: Future studies could focus on further characterizing the essential tensors in GR^h-RF_n beyond their non-vanishing properties, aiming to provide more in-depth geometric interpretations or classification results.

Interdisciplinary Research: Given the theoretical importance of recurrent spaces in Finsler geometry, interdisciplinary research involving physics, particularly in fields like general relativity and quantum mechanics, may offer fruitful applications of the theoretical results derived here.

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الفضاءات المعممة -R^h التكرارية في هندسة فنسلر: الخصائص والمتطابقات والتوصيفات

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الملخص: في هذه الورقة البحثية، ندرس فئة من فضاءات فنسلر، التي تُسمى الفضاءات المعممة -R^h التكرارية المشار إليها بـ(GR^h-RF_n)، حيث يحقق موتر الانحناء الثالث لِكارتان شرطًا معينًا يتضمن حقول متجهات غير معدومة. نقدم عدة توصيفات وخصائص لهذه الفضاءات، بدءًا من تكافؤ شكلين متميزين لشرط الانحناء، ونثبت بعض النظريات الأساسية حول عدم زوال الموترات الأساسية مثل موتر ريتشي، متجه الانحناء، موتر الانحراف، والمقياس الانحنائي. علاوة على ذلك، نشتق متطابقات هامة مرتبطة بالتفاضل المتغاير للموترات في هذه الفضاءات، مثل المشتقة المتغايرة -h لموتر-h(v) الالتواء وموتر الانحراف. تلعب هذه المتطابقات دورًا حاسمًا في فهم البنية الهندسية للفضاءات GR^h-RF_n ، وتساهم النتائج في توسيع النظرية العامة للفضاءات التكرارية في هندسة فنسلر. تختتم الورقة بتقديم عدة متطابقات جديدة تنطبق في -GR^h RF_n، مما يعزز الإطار النظري للفضاءات التكرارية العامة في الهندسة التفاضلية.

كلمات مفتاحيه: فضاء فنسلر، تعميم فضاء فنسلر R^h - أحادي المعاودة ، موثر الالتواء.