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The New Technique of Adomian Method for Singular BVPs in Some Third and nth Order Ordinary Differential Equations

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Abstract:

In this paper, we present a fast and accurate numerical technique to solve some third and nth order Ordinary Differential Equations(ODEs). The Adomian Decomposition Method (ADM) with a new differential operator are applied for solving Singular Boundary Value Problems (SBVPs). The effectiveness of the novel approach is verified by several linear and non-linear examples.

Keywords: Adomian decomposition method; singular boundary value problems; third and nth order differential equations

Introduction: The development of efficient and accurate methods for solving SBVPs has been a subject of extensive research in the field of applied mathematics. One such method, which has garnered significant attention in recent years, is the adomian decomposition method (ADM) [1,2]. This method provides a powerful framework for solving a wide range of differential equations, including higher-order equations. Recent advancements in the field of solving higher order equations have led to the development of a new differential operator that combines the ADM with other techniques. This innovative approach offers a promising solution for efficiently solving higher order differential equations, which often present challenges due to their complex nature. The ADM has been widely used in solving linear and non-linear problems [3,4]. Additionally, recent studies have applied the ADM to solve fifth [12] and higher order ordinary differential equations (ODEs) [7]. Moreover, the modified ADM has been utilized in solving higher order ODEs [5,8]. This research draws inspiration from the aforementioned studies and proposes the use of a new differential operator within the provide more accurate and efficient solutions. This paper aims to contribute to the existing framework of ADM for finding solutions to third and nth order SBVPs of ODEs as shown in form (1) and (31). This approach aims to overcome the limitations of existing methods and body of knowledge by offering an innovative and reliable method.

Third Order SBVPs of ODEs:

Let's consider a third-order differential equation as shown in the following form

$$y''' + \frac{2a+b}{x} y'' + \frac{(a-2b)(a-1)}{x^2} y' - \frac{b(a-1)}{x^3} y + h(x,y) = g(x), \tag{1}$$

where x ∈ [0,j], where j∈R , j≠0, and a,b are real numbers, h(x,y) and g(x) are real functions. The ADM is a powerful technique used to solve a wide range of differential equations in various fields of science and engineering. The use of ADM has been utilized to solve fifth boundary value problems [12]. The modified ADM has been developed by researchers like Hasan and Wazwaz to improve the accuracy and reliability of the method [7,10]. This modification has proven to be particularly useful in solving sixth and higher-order boundary value problems [8,11,13]. In this section, we propose a new technique of the ADM to solve SBVPs of a family of third-order. This technique combines the ADM with reflection-capable a new differential operator, which use to solve third-order SBVPs of ODE in the form (1). So, we suggest that Eq.(1) is derived using the differential operator L, which is given as shown in the form below:

$$L(.) = x^{-a} \frac{d^2}{dx^2} x^{a+b} \frac{d}{dx} x^{-b}(.), \tag{2}$$

where a >2, b ∈ [0,2], and its inverse integration is given as follows:

$$L^{-1}(.) = x^{b} \int_j^x x^{-(b+a)} \int_0^x \int_0^x x^{a}(.)dx\, dx\, dx. \tag{3}$$

In a new operator form L, Eq. (1) can be written as follows:

Ly=g(x)-h(x,y). (4)

2.1. The Solution Algorithm by A New Technique

In this part, we discuss the use of ADM with the proposed differential operator to find numerical solutions for the equation in the form (4) with the boundary conditions given $y(0) = c_0$, y'(0)=c_1 , y(j)=p, where c_0, c_1, p are constants.

Applying L^{-1} to Eq. (4) leads to:

$$L^{-1}(Ly) = L^{-1}g(x) - L^{-1}h(x,y), \tag{5}$$

the left-hand side of Eq. (5) become

$$L^{-1}(Ly) = x^{b} \int_j^x x^{-b-a} \int_0^x \int_0^x x^{a} [y''' + \frac{2a+b}{x} y'' + \frac{(a-2b)(a-1)}{x^2} y' - \frac{b(a-1)}{x^3} y\ ] dxdxdx,$$

$$= x^{b} \int_j^x x^{-a-b} \int_0^x\ [x^{a} y''(x) + (a-b)x^{a-1} y'(x) - b(a-1)x^{a-2} y(x) - 0\ ] dxdx,$$

$$=x^{b}\int_{j}^{x}x^{-a-b}\left[x^{a}y'(x)-bx^{a-1}y(x)-0\right]dx$$

$$=x^{b}\int_{j}^{x}x^{-a-b}\left[x^{a}y'(x)-bx^{a-1}y(x)\right]dx$$

$$=x^{b}\left[x^{-b}y(x)-pj^{-b}\right]$$

$$=y(x)-pj^{-b}x^{b}$$, then

$$L^{-1}(Ly)=y(x)-j^{-b}px^{b},$$

So, Eq. (5) becomes

$$y(x)-pj^{-b}x^{b}=L^{-1}g(x)-L^{-1}h(x,y), \tag{6}$$

which is equivalent

$$y(x)=pj^{b}x^{b}+L^{-1}g(x)-L^{-1}h(x,y), \tag{7}$$

where $L(x^{b})=0$,

to ADM assuming that the unknown function y(x), which represents the solution to the equation, and the non-linear variable h(x,y) can be represented by infinite series as follows [7-9]:

$$y(x)=\sum_{n=0}^{\infty}y_{n}(x), \tag{8}$$

and

$$h(x,y)=\sum_{n=0}^{\infty}A_{n}, \tag{9}$$

by substituting Eq. (8) and Eq. (9) into Eq. (7), we have

$$\sum_{n=0}^{\infty}y_{n}=pj^{-b}x^{b}+L^{-1}g(x)-L^{-1}\sum_{n=0}^{\infty}A_{n}, \tag{10}$$

where the n-th component of the solution y(x), is obtained iteratively [1,2], and A_n are the Adomain polynomial, and its general formulation can be seen in references [1,9], which are obtained in the following formula

$$A_{n}=\frac{1}{n!}\frac{d^{n}}{d\lambda^{n}}\left[F\left(\sum_{i=0}^{\infty}\lambda^{i}y_{i}\right)\right]_{\lambda=0}, n=0,l,\ldots$$ Where λ is a parameter. From this, we obtain

$$A_{0}=F\left(y_{0}\right),$$

$$A_{1}=y_{1}F'(y_{0}),$$

$$A_{2}=y_{2}F'(y_{0})+\frac{1}{2}y_{1}^{2}F''(y_{0}),$$

$$A_{3}=y_{3}F'(y_{0})+y_{1}y_{2}F''(y_{0})+\frac{1}{3!}y_{1}^{3}F'''(y_{0}),$$

$$\ldots \tag{11}$$

From Eq.(10), we obtain the iterative relationship of the solution from the point of view of ADM as follows :

$$y_0 = pj^{-b}x^b + L^{-1}g(x),$$
$$y_{n+1} = -L^{-1}A_n, n \geq 0, \tag{12}$$

which gives

$$y_0 = pj^{-b}x^b + L^{-1}g(x),$$
$$y_1 = -L^{-1}A_0,$$
$$y_2 = -L^{-1}A_1,$$
$$y_3 = -L^{-1}A_2, \ldots.$$

Now, from the iterative relationship in Eq.(12), we obtain a series of solution, then we obtain the approximate solution to y in Eq.(7). In addition, and for numerical reasons, we can use the n-term approximate [8]

$$\varphi_n = \sum_{k=0}^{n-1} y_k(x).$$

Which can be used to approximate the exact solution. The convergent of this series has proved in [1,6].

2.2. Applications

In this part, we provide practical applications to demonstrate the validity and accuracy of the proposed method in finding numerical solutions of third order SBVPs of linear and non linear ODEs.

Example 1. We consider the non-linear boundary value problem:

$$y''' + \frac{11}{2x}y'' + \frac{4}{x^2}y' - \frac{1}{x^3}y = x - y^2, \ x \in [0,1], \tag{13}$$

$y(0) = 0$, $y'(0) = 0$, $y(1) = 1$, with exact solution $y(x) = x^{0.5}$.

In a new operator, Eq.(13) can be written as

$$Ly = x - y^2, \tag{14}$$

where the differential operator as

$$L(.) = x^{-3}\frac{d^2}{dx^2}x^{\frac{7}{2}}\frac{d}{dx}x^{-0.5}(.),$$

and the inverse operator

$$L^{-1}(.) = x^{0.5}\int_1^x x^{\frac{-7}{2}}\int_0^x\int_0^x x^3(.)dxdxdx,$$

by applying L^{-1} on both sides of Eq.(14), and using the conditions, yields

$$y(x)-x^{0.5}=L^{-1}x-L^{-1}y^{2}, \tag{15}$$

substituting the decomposition series y_n for y(x), and A_n for the non linear term y^2 into Eq.(15) gives

$$\sum_{n=0}^{\infty} y_n(x)=x^{0.5}+L^{-1}x-L^{-1}\sum_{n=0}^{\infty} A_n, \tag{16}$$

from Eq.(16), we obtain the iterative relationship of the solution from the point of view of ADM as follows:

$$y_0=x^{0.5}+L^{-1}x,$$
$$y_{n+1}=-L^{-1}A_n, n\geq 0, \tag{17}$$

where Adomain polynomials A_n for the non linear term y^2 are given by

$$A_0=y_0^2,$$
$$A_1=2y_1y_0,$$
$$A_2=y_1^2+2y_0y_2, \ldots$$

Using Eq. (17), the first several calculated solution components are

$$y_0=x^{0.5}+\frac{x^7}{195},$$
$$y_1=-\frac{x^7}{195}-\frac{8x^6}{57915}-\frac{x^{16.5}}{12320100},$$

the series solution by ADM is given by

$$y(x)=y_0+y_1=x^{0.5}-\frac{8x^6}{57915}-\frac{x^{16.5}}{12320100},$$

that converges to the exact solution $y(x)=x^{0.5}$.

Table 1. The comparison of numerical error between the exact solution and the approximate solution by ADM

X	Exact solution	ADM	Absoulte Error
0.0	0.000000000000	0.000000000000	0.000000000000
0.1	0.3162277660168	0.3162277658787	0.0000000001381
0.2	0.447213595500	0.4472135866594	0.0000000088405
0.3	0.5477225575052	0.5477224568059	0.0000001006993
0.4	0.6324555320387	0.632454966239	0.0000005657947
0.5	0.7071067811865	0.707106228502	0.0000021583364
0.6	0.7745966692415	0.7745902244685	0.0000006444773
0.7	0.8366600265341	0.8366437750436	0.0000162514905
0.8	0.8944271909999	0.894390780957	0.0000362129042
0.9	0.9486832980505	0.9486098739915	0.0000734024059
1.0	1.000000000000	0.9996817853604	0.0001382146396

Table 1 show ion and the approximate sols the error of the exact solut by ADM of problem (13). In this example, we only use the forest two terms to approximate the exact solution. absolute error is very small, the From the error column we can find that the ADM with a new differential operator has a high convergence order. The more terms we use, the higher accuracy we get.

Example 2. We consider the non-linear boundary value problem:

$$y'''+\frac{5.7}{x}y''+\frac{4.8}{x^2}y'-\frac{0.6}{x^3}y=x^{0.09}-y^2, \qquad x\in[0,1], \tag{18}$$

$$y(0)=0, y'(0)=0, y(1)=1,$$

with exact solution $y(x)=x^{0.3}$.

In an operator Eq. (18) can be written as

$$Ly=x^{0.09}-y^2, \tag{19}$$

where the differential operator as

$$L(.)=x^{-3}\frac{d^2}{dx^2}x^{3.3}\frac{d}{dx}x^{-0.3}(.),$$

and the inverse operator

$$L^{-1}(.)=x^{0.3}\int_1^x x^{-3.3}\int_0^x\int_0^x x^3(.)dxdxdx,$$

by applying L^{-1} on both sides of Eq.(19), and using the conditions, yields

$$y(x)=x^{0.3}+L^{-1}x^{0.09}-L^{-1}y^2, \tag{20}$$

where $L(x^{0.3})=0$, substituting the decomposition series y_n for y(x), and A_n for the non linear term into Eq.(20) gives

$$\sum_{n=0}^{\infty}y_n(x)=x^{0.3}+L^{-1}x^{0.09}-L^{-1}\sum_{n=0}^{\infty}A_n. \tag{21}$$

From Eq.(21), we obtain the iterative relationship of the solution from the point of view of ADM as follows:

$$y_0=x^{0.3}+L^{-1}(x^{0.09}),$$

$$y_{n+1}=-L^{-1}A_n, n\geq 0, \tag{22}$$

where Adomain polynomials A_n for the non linear term y^2 are given by

$$A_0=y_0^2,$$

$$A_1=2y_1y_0,$$

$$A_2=y_1^2+2y_0y_2,\ \ldots$$

Using Eq.(22), the first several calculated solution components are

$$y_0=x^{0.3}+\frac{x^{3.97}}{2.769413},$$

$$y_1=-\frac{x^{3.97}}{2.769413}-\frac{2x^{4.97}}{459.235927734969}-\frac{x^{18.4609}}{37529.9904503692233},$$

the series solution by ADM is given by

$$y(x)=y_0+y_1=x^{0.3}-\frac{2x^{4.97}}{459.235927734969}-\frac{x^{18.4609}}{37529.9904503692233},$$

that converges to the exact solution $y(x)=x^{0.3}$.

Example 3. We consider the non-linear boundary value problem:

$$y'''+\frac{11.4}{x}y''+\frac{24}{x^2}y'-\frac{12}{x^3}y=x^{0.36}-y^2,\qquad x\in[0,1], \tag{23}$$

$$y(0)=0, y'(0)=0, y(1)=1,$$

with exact solution $y(x)=x^{0.6}$.

Eq. (23) can be written as

$$Ly=x^{0.36}-y^2, \tag{24}$$

where the differential operator as

$$L(.)=x^{-6}\frac{d^2}{dx^2}x^{6.6}\frac{d}{dx}x^{-0.6}(.),$$

and the inverse operator

$$L^{-1}(.)=x^{0.6}\int_1^x x^{-6.6}\int_0^x\int_0^x x^6(.)dx\,dx\,dx\,,$$

by applying L^{-1} on both sides of Eq.(24), and using the conditions, yields

$$y(x)=x^{0.6}+L^{-1}x^{0.36}-L^{-1}y^2, \tag{25}$$

substituting the decomposition series y_n for y(x), and A_n for the non linear term y^2 into Eq. (25) gives

$$\sum_{n=0}^{\infty}y_n(x)=x^{0.6}+L^{-1}x^{0.36}-L^{-1}\sum_{n=0}^{\infty}A_n, \tag{26}$$

from Eq.(26), we obtain the iterative relationship of the solution from the point of view of ADM as follows:

$$y_0=x^{0.6}+L^{-1}(x^{0.36}),$$

$$y_{n+1}=-L^{-1}A_n, n\ge 0, \tag{27}$$

where A_n for the non linear term y^2 are given by

$$A_0=y_0^2,$$

$$A_1=2y_1y_0,$$

$$A_2=y_1^2+2y_0y_2, \ldots$$

Using Eq. (27), the first several calculated solution components are

$$y_0 = x^{0.6} + \frac{x^{3.36}}{169.821696},$$

$$y_1 = -\frac{x^{3.36}}{169.821696} - \frac{2x^{5.016}}{67722.10635428698} - \frac{x^{9.72}}{362305207.5174},$$ the series solution by ADM is given by

$$y(x) = y_0 + y_1 = x^{0.6} - \frac{2x^{5.016}}{67722.10635428698} - \frac{x^{9.72}}{362305207.5174},$$

that converges to the exact solution $y(x) = x^{0.6}$.

3. nth-Order SBVPs of ODEs

Many studies have used the ADM to solve BVPs for certain types of ODEs of different orders, see [7]. Other research has also modified the ADM to solve SBVPs for other types of higher order ODEs, see [8,11]. In this section, we propose a new technique for the ADM to solve SBVPs for some types of nth order ODEs. This technique combines the ADM with a new differential operator of order n that can be inverted to an integral operator. Firstly, we use the new differential operator to derive the ODEs of different orders according to the value of n. Secondly, we discuss the solution algorithm using the ADM with the new operator and its inverse. Finally, we present practical examples of the proposed method to demonstrate its effectiveness and efficiency in solving SBVPs.

3.1. General Derivation of nth -Order ODEs

We consider the equation in form

$$Ly + h(x,y) = g(x), \tag{28}$$

where L is a linear operator, h(x,y) and g(x) are real functions, we use Eq.(1), to derive higher order differential equation and we suggest L is a differential operator as below:

$$L(.) = x^{-n} \frac{d^{n-1}}{dx^{n-1}} x^{n+m} \frac{d}{dx} x^{-m}(.), \tag{29}$$

where $n \ge m > 0, n \ge 1.$

In a new operator form L, the Eq.(1), we can rewrite as:

$$x^{-n} \frac{d^{n-1}}{dx^{n-1}} x^{n+m} \frac{d}{dx} x^{-m}(y) + h(x,y) = g(x). \tag{30}$$

To find different equations of higher order, we vary the value of n.

When we set n = 1 in Eq. (30), we obtain

$$y' - \frac{m}{x} y + h(x,y) = g(x),$$

when we set n = 2 in Eq. (30), we obtain

$$y'' + \frac{(2-m)}{x} y' - \frac{m}{x^2} y + h(x,y) = g(x),$$

when we set n=3 in Eq.(30), we obtain

$$y''' + \frac{(6-m)}{x}y' - \frac{(6-4m)}{x^2}y - \frac{2m}{x^3} + h(x,y) = g(x),$$

when we set n = 4 in Eq. (30), we obtain

$$y^{(4)} + \frac{(12-m)}{x}y^{(3)} + \frac{(36-9m)}{x^2}y'' + \frac{(24-18m)}{x^3}y' - \frac{6m}{x^4}y + h(x,y) = g(x),$$

by continuing with the same procedure gives us the following generalization:

$$\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{n!}{(n-r)!}x^{-r}y^{(n-r)} - m\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{(n-1)!}{(n-r-1)!}x^{-r-1}y^{(n-r-1)} = g(x) - h(x,y). \tag{31}$$

3.2 The Solution Algorithm by A New Technique

In this part, we discuss the use of ADM with the proposed differential operator to find numerical solutions for the equation in form (31), with boundary conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, ..., y^{(n-2)}(0) = c_{n-2}, y(j) = k,$$

$0 \le x \le j$, where j∈R, j≠0, and $c_0, c_1, ..., c_{n-2}, k$, are real constants. According to the ADM we can rewrite Eq. (31) in the form as:

$$Ly = g(x) - h(x,y), \tag{32}$$

and the inverse operator L^{-1} as:

$$L^{-1}(.) = x^m \int_j^x x^{-(m+n)} \int_0^x ... \int_0^x \int_0^x x^n (.) dx dx dx ... dx dx, \tag{33}$$

by applying L^{-1} on Eq. (32), we obtain

$$y(x) = \phi(x) - L^{-1}(g(x) + h(x,y)), \tag{34}$$

where $\phi(x)$ is obtained as the result of boundary conditions.

The ADM represents the solution y(x), and the non-linear function h(x,y) by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad \text{and} \tag{35}$$

$$h(x,y) = \sum_{n=0}^{\infty} A_n, \tag{36}$$

where the elements $y_n(x)$ of the solution y(x) will be determined repeatable [1,2], and the A_n are the Adomain polynomials,

which are obtained in the following formula [1,3]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[F(\sum_{i=0}^{\infty}\lambda^i y_i)]_{\lambda=0}, n = 0, l, ... \tag{37}$$

where λ is a parameter. From Eq. (37), we have

$$A_0 = F(y_0),$$

$$A_1 = y_1 F'(y_0),$$

$$A_2 = y_2 F'(y_0) + \frac{1}{2} y_1^2 F''(y_0),$$

$$A_3 = y_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{1}{3!} y_1^3 F'''(y_0),$$

$$.... \tag{38}$$

Substituting Eq. (35) and Eq. (36) into Eq. (34), we have

$$\sum_{n=0}^{\infty} y_n = \phi(x) + L^{-1} g(x) - L^{-1} \sum_{n=0}^{\infty} A_n, \tag{39}$$

according to the ADM, all terms of y exact y_0 are determined by recursive relation, that is [2]

$$y_0 = \phi(x) + L^{-1} g(x),$$

$$y_{n+1} = -L^{-1} A_n, n \geq 0, \tag{40}$$

which gives

$$y_0 = \phi(x) + L^{-1} g(x),$$

$$y_1 = -L^{-1} A_0,$$

$$y_2 = -L^{-1} A_1,$$

$$y_3 = -L^{-1} A_2, ...$$

Using the Eq.(40), we can determine the components y_n, and therefore, we can immediately obtain series solutions of y(x) in Eq.(34). In addition, and for numerical reasons, we can use the n-term approximate [8]

$\varphi_n = \sum_{k=0}^{n-1} y_k(x)$. Which can be used to approximate the exact solution. The convergent of this series has proved in [1,6].

3.3 Testing the Method with Examples

In this part, we provide practical applications to demonstrate the validity and accuracy of the proposed method in finding numerical solutions of nth order SBVPs of linear and non linear ODEs.

Example 1. We assume the non-linear boundary value problem:

$$y'' - \frac{2}{x^2} y = x^4 - y^2, \quad 0 \leq x \leq 1, \quad y(0) = 0, y(1) = 1, \tag{41}$$

with exact solution $y(x) = x^2$.

Eq. (41) which can be written as:

$$Ly = x^4 - y^2, \tag{42}$$

where the differential operator as:

$L(.) = x^{-2} \frac{d}{dx} x^4 \frac{d}{dx} x^{-2}(.)$, and the inverse operator

$$L^{-1}(.) = x^2 \int_1^x x^{-4} \int_0^x x^2 (.) dx dx,$$

by applying L^{-1} on both sides of Eq.(42), and using the conditions, yield

$$y(x) = x^2 + L^{-1}x^4 - L^{-1}y^2, \tag{43}$$

substituting the decomposition series y_n for y(x) and A_n for the non-linear term y^2 into Eq.(43) gives

$$\sum_{n=0}^{\infty} y_n(x) = x^2 + L^{-1}x^4 - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{44}$$

from Eq.(44), we obtain the iterative relationship of the solution from the point of view of ADM as follows:

$$y_0 = x^2 + L^{-1}x^4,$$

$$y_{n+1} = -L^{-1}A_n, n \geq 0, \tag{45}$$

where the non linear term y^2 has the first few Adomain polynomials A_n are given by

$$A_0 = y_0^2,$$

$$A_1 = 2y_1y_0,$$

$$A_2 = y_1^2 + 2y_0y_2, \ldots$$

Using Eq.(45), the first several calculated solution components are

$$y_0 = x^2 + \frac{x^6}{28},$$

$$y_1 = -\frac{x^6}{28} - \frac{x^{10}}{1232} - \frac{x^{14}}{141120},$$

$$y_2 = \frac{x^{10}}{1232} + \frac{72x^{14}}{141120} + \frac{188x^{18}}{21450240} + \frac{x^{22}}{908812800},$$

the series solution by ADM is given by

$$y(x) = y_0 + y_1 + y_2 = x^2 + \frac{71x^{14}}{141120} + \frac{188x^{18}}{21450240} + \frac{x^{22}}{908812800},$$

that converges to the exact solution $y(x) = x^2$.

Table 2. The comparison of numerical error between the exact solution and the approximate solution by ADM.

X	Exact solution	ADM	Absoulte Error
0.0	0.00	0.0000000000000	0.000000000000
0.1	0.01	0.0100000000000	0.000000000000
0.2	0.04	0.0400000000001	0.0000000000001
0.3	0.09	0.0900000000241	0.0000000000241
0.4	0.16	0.1600000013506	0.0000000013506
0.5	0.25	0.2500000307113	0.0000000307113
0.6	0.36	0.3600003943545	0.0000003943545
0.7	0.49	0.4900003957124	0.0000003957124
0.8	0.64	0.6400221433845	0.0000221433845
0.9	0.81	0.8101152307982	0.0001152307982
1.0	1.00	01905040081122	0.0005040081122

Table 2 In this example, .shows the error of the exact solution and the approximate solution we only use the forest three terms to approximate the exact solution. From the error column we can find that the absolute error is very small, the ADM with a new differential operator use, the higher accuracy we get has a high convergence order. The more terms we.

Example 2. We consider the non-linear boundary value problem:

$$y''' + \frac{11}{2x}y'' + \frac{4}{x^2}y' - \frac{1}{x^3}y = x - y^2, \qquad x \in [0,1], \tag{46}$$

$y(0)=0, y'(0)=0, y(1)=1,$ with exact solut $y(x)=x^{0.5}$.

Eq.(46) which can be written as:

$$Ly = x - y^2, \tag{47}$$

where the differential operator as:

$$L(.) = x^{-3}\frac{d^2}{dx^2}x^{\frac{7}{2}}\frac{d}{dx}x^{-0.5}(.),$$

and the inverse operator

$$L^{-1}(.) = x^{0.5}\int_1^x x^{\frac{-7}{2}}\int_0^x\int_0^x x^3(.)dxdx\,,$$

by applying L^{-1} on both sides of Eq.(47), and using the conditions, yield

$$y(x) = x^{0.5} + L^{-1}x - L^{-1}y^2, \tag{48}$$

substituting the decomposition series y_n for y(x) and A_n for the non-linear term y^2 into Eq. (48) gives

$$\sum_{n=0}^{\infty} y_n(x) = x^{0.5} + L^{-1}x - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{49}$$

from Eq.(49), we obtain the iterative relationship of the solution from the point of view of ADM as follows:

$$y_0 = x^{0.5} + L^{-1}x\ ,$$

$$y_{n+1} = -L^{-1}A_n, n \ge 0, \tag{50}$$

where the non linear term y^2 has the first few Adomain polynomials A_n are given by

$$A_0 = y_0^2,$$

$$A_1 = 2y_1y_0,$$

$$A_2 = y_1^2 + 2y_0y_2, \ldots$$

Using Eq.(50), the first several calculated solution components are

$$y_0 = x^{0.5} + \frac{x^7}{195},$$

$$y_1 = -\frac{x^7}{195} - \frac{8x^6}{57915} - \frac{x^{16.5}}{12320100},$$

the series solution by ADM is given by that converges to the exact solution $y(x)=x^{0.5}$.

Example 3. We consider the linear boundary value problem:

$$y''''+\frac{11}{x}y'''+\frac{27}{x^2}y''+\frac{6}{x^3}y'-\frac{6}{x^4}y=g(x)-xy,\ x\in[0,1], \tag{51}$$

$$y(\frac{11}{7})=0, y'(0)=0, y''(0)=-1, y'''(0)=0,$$

with exact solution $y=\cos x$, where

$$g(x)=\frac{x^5cosx+x^4cosx+11x^3sinx-27x^2cosx-6xsinx-6cosx}{x^4},$$

the Eq.(51) which can be written as:

$$Ly=g(x)-xy, \tag{52}$$

where the differential operator as:

$L(.)=x^{-4}\frac{d^3}{dx^3}x^5\frac{d}{x}x^{-1}(.)$, and the inverse operator is given by:

$$L^{-1}(.)=x\int_{\frac{11}{7}}^{x}x^{-5}\int_0^x\int_0^x\int_0^x x^4(.)dxdxdxdx,$$

by applying L^{-1} on both sides of Eq.(52), and using the conditions, yields

$$y(x)=L^{-1}g(x)-L^{-1}xy, \tag{53}$$

substituting the decomposition series y_n for y(x) into Eq. (53) gives

$$\sum_{n=0}^{\infty}y_n(x)=L^{-1}g(x)-L^{-1}xy, \tag{54}$$

from Eq.(54), we obtain the iterative relationship of the solution from the point of view of ADM as follows:

$$y_0=L^{-1}g(x),$$

$$y_{n+1}=-L^{-1}xy_n, n>0, \tag{55}$$

Using Eq. (55), the first several calculated solution components are

$$y_0=cosx+\frac{x^4}{1344}+\frac{x^6}{8640}+...,$$

$$y_1=-\frac{x^4}{1344}-\frac{x^6}{8640}-...-\frac{x^{10}}{20756736}-\frac{x^{12}}{259452900}-...,$$

$$y_2=\frac{x^{10}}{20756736}+\frac{x^{12}}{259452900}+...+\frac{x^{15}}{1422749712384}+\frac{x^{17}}{28395214848000}+...$$

The series solution by ADM is given by

$$y(x)=y_0+y_1+y_2=cosx+\frac{x^{15}}{1422749712384}+\frac{x^{17}}{28395214848000}+...$$

That converges to the exact solution $y(x)=cosx$.

4 Conclusion

In this paper, we successfully solved some third and n^{th} order SBVPs of ODEs by combining the ADM with a new differential operator. The method facilitates the solution process and offers practical and accurate results. The results obtained from various test cases validate the accuracy and efficiency of our proposed approach, which offers a powerful and innovative solution for solving these problems. This paper provides a solid foundation for further exploration and development in this area.

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التقنية الجديدة لطريقة أدوميان لأجل مشاكل القيمة الحدية الشاذة في المعادلات التفاضلية العادية من الرتبة الثالثة والرتبة ن

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الملخص: في هذه الورقة ، تقدم تقنية عددية سريعة ودقيقة لحل بعض المعادلات التفاضلية العادية من الرتبة الثالثة والرتبة ن . يتم تطبيق طريقة أدوميان التحليلية مع مؤثر تفاضلي جديد لحل مشاكل القيمة الحدية الشاذة ،يتم التحقق من فعالية النهج الجديد من خلال عدد من الأمثلة الخطية وغير الخطية.

الكلمات المفتاحية:
طريقة أدوميان التحليلية ، مشاكل القيمة الحدية الشاذة ، المعادلات التفاضلية العادية من الرتبة الثالثة والرتبة ن.