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On Pseudo T-Trirecurrent Finsler Space in Berwald Sense

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Abstract: In this present paper, we introduce a Finsler space which pseudo curvature tensor satisfies the trirecurrence property in sense of Berwald. Certain identities belong to this space have been studied. Finally, the trirecurrence property in a projection on indicatrix with respect to Berwald connection has been discussed

Keywords: pseudo T-Trirecurrent, flat pseudo T-Trirecurrent, projection on indicatrix.

1.Introduction.

The recurrence property has been studied by the Finslerian geometrics. Sinha [16] introduced the torsion tensor T^i_{jk} and curvature tensor T^i_{jkh} from the division tensor T^i_j. Dabey and Singh [2] and Pandey and Dwivedi [5,6] considered the space equipped curvature tensor T^i_{jkh} is recurrent and called it T-recurrent Finsler space. They also considered there in projectively flat-T-recurrent space and obtained certain results belong to these spaces. Saleem [11] studied the flat of recurrent curvature tensor fields in Finsler space. Further, Qasem and Saleem [9], Pandey and Verma [7], Saleem and Abdallah [14], Singh [15] and Sinha [16] were studied on birecurrent curvature tensor fields in Finsler space. Saleem and Abdallah [13] study the projection on indicatrix for some tensors whose satisfy the birecurrence property. Saleem and Abdallah [13] considered the space equipped curvature tensor T^i_{jkh} is birecurrent and called it T-birecurrent Finsler space. They also considered there in projectively flat-T-recurrent space and obtained certain results belong to these spaces. Let us consider an n-dimensional Finsler space F_n equipped with the line elements (x, y) and the fundamental metric function F is positively homogeneous of degree one in y^i.

Berwald's covariant derivative of the vector y^j vanish identically. i. e.,

(1.1) a) $\beta_l y^j = 0$, b) $\beta_l l^j = 0$,

c) $\beta_l F = 0$ and d) $l^j = \frac{y^j}{F}$.

Definition 1.1. The projection of any tensor T^i_j on indicatrix is given by [3]

(1.2) $p.T^i_j = T^\alpha_\beta\, h^i_\alpha\, h^\beta_j$,

where the angular metric tensor is homogeneous function of degree zero in y^i and defined by

(1.2) $h^i_j := \delta^i_j - l^i l_j$.

Definition 1.2. If the projection of a tensor T^i_j on indicatrix I_{n-1} is the same tensor T^i_j, the tensor is called an indicatrix tensor or an indicatory tensor.

In flat pseudo T-birecurrent space, the curvature tensor H^i_{jkh}, torsion tensor H^i_{kh}, division tensor H^i_h , Ricci tensor H_{jk}, curvature vector H_k and curvature scalar H are birecurrent ,i.e. [13]

(1.3) $\beta_m\beta_l H^{\ i}_{jkh} = a_{lm} H^{\ i}_{jkh}$;

(1.4) $\beta_m\beta_l H^{\ i}_{kh} = a_{lm} H^{\ i}_{kh}$;

(1.5) $\beta_m\beta_l H^{\ i}_{h} = a_{lm} H^{\ i}_{h}$;

(1.6) $\beta_m\beta_l H_{kh} = a_{lm} H_{kh}$;

(1.7) $\beta_m\beta_l H_h = a_{lm} H_h$;

(1.8) $\beta_m\beta_l H = a_{lm} H$.

2. PRELIMINARIES

In this section, we introduce some important concepts and definitions.

The pseudo division tensor T^i_j is positively homogeneous of degree 2 in y^i and defined by [16]:

(2.1) $T^i_j = -\{H\delta^i_j + \frac{1}{n+1}(\dot\partial_r H^r_j - \dot\partial_j H) y^i\}$.

The pseudo torsion tensor T^i_{jk} is positively homogeneous of degree 1 in y^i and defined by:

(2.2) $T^i_{jk} = \frac{1}{n+1}\{y^i \mathrm{H}^r_{rkj} + 2\delta^i_{[j}(H_{k]} + \dot\partial_{k]} H)\}$.

The pseudo curvature tensor T^i_{jkh} is positively homogeneous of degree 0 in y^i and defined by:

(2.3) $T^i_{jkh} = \frac{1}{n+1}\{\delta^i_{[j}\mathrm{H}^r_{k]hr} + y^i\dot\partial_j \mathrm{H}^r_{rkh} + 2\delta^i_{[k}(H_{j]h} + \dot\partial_h\dot\partial_j H)\}$.

From tensor S. P. Sinha [16] obtained the tensors as follows:

(2.4) $a)\ \ T^i_{jk} = \frac{2}{3}\dot\partial_{[j} T^i_{k]}$

and $b)\ \ T^i_{jkh} = \dot\partial_j T^i_{kh}$.

The curvature tensor T^i_{jkh} and torsion tensor T^i_{kh} satisfies the following identities

(2.5) $a)\ T^i_{jkh} y^j = T^i_{kh}$ and $b)\ T^i_{kh} y^k = T^i_h$.

Also, this tensors satisfies the following:

(2.6) $a)\ T^i_{jkh} = W^i_{jkh} - H^i_{jkh}$,

$b)\ T^i_{kh} = W^i_{kh} - H^i_{kh}$ and $c)\ T^i_k = W^i_k - H^i_k$.

The curvature tensor H^i_{jkh}, the torsion tensor H^i_{jk} and the division tensor H^i_j are positively homogeneous of degree zero, one and two in y^i, respectively. The curvature tensor H^i_{jkh} is skew -symmetric in their lower indices satisfy the following:

(2.7) $a)\ \mathrm{H}^i_{jkh} y^j = H^i_{kh}$, $b)\ \mathrm{H}^i_{kh} y^k = H^i_h$,

$c)\ \ H^i_{jki} = H_{jk}$, $d)\ H_{jk} y^j = H_k$,

$e)\ \ H_k y^k = (n-1)H$,

$g)\ \ H^i_i = (n-1)H$

The Bianchi identities for curvature tensor H^i_{jkh} is given by

$l)\ \ H^i_{jkh} + H^i_{khj} + H^i_{hjk} = 0$.

The projective curvature tensor W^i_{jkh}, the torsion tensor W^i_{jk} and the division tensor W^i_j are positively homogeneous of degree zero, one and two in y^i, respectively. And satisfy the following: [15]

(2.8) $a)\ W^i_{jkh} y^j = W^i_{kh}$, $b)\ W^i_{kh} y^k = W^i_h$

and $c)\ W^i_{jki} = 0$.

The Bianchi identities for projective curvature tensor field W^i_{jkh} is given by [15]

(2.9) $W^i_{jkh} + W^i_{khj} + W^i_{hjk} = 0$.

Taking skew – symmetric part of (2.6a) with respect to j, k and h, using (2.7l) and (2.9), we get

(2.10) $T^i_{jkh} + T^i_{khj} + T^i_{hjk} = 0$.

A Finsler space is called a pseudo T-recurrent Finsler space, if it's the curvature tenser T^i_{jkh} satisfies ([5], [6])

(2.11) $\beta_l T^i_{jkh} = \lambda_l T^i_{jkh}$, $T^i_{jkh} \neq 0$,

where λ_l is non-zero covariant vector field.

Since Finsler space is projective flat, we have [15]

(2.12) $a)\ W^i_{jkh} = 0$, $b)\ W^i_{kh} = 0$,

and $c)\ W^i_j = 0$.

A Finsler space is called a pseudo T-birecurrent Finsler space, if it's the curvature tenser T^i_{jkh} satisfies [13]:

(2.13) $\beta_m \beta_l T^{\ i}_{jkh} = a_{lm} T^i_{jkh}$, $T^i_{jkh} \neq 0$,

where a_{lm} recurrence covariant tensor field of second order

(2.14) $\beta_m \beta_l T^{\ i}_{kh} = a_{lm} T^i_{kh}$

(2.15) $\beta_m \beta_l T^{\ i}_{h} = a_{lm} T^i_{h}$

The curvature tensor T^i_{jkh} of a pseudo T-birecurrent Finsler space on indicatrix is birecurrent in sense of Berwald [13]

(2.16) $\beta_m \beta_l (p\ T^i_{jkh}) = a_{lm} (p\ .T^i_{jkh})$.

(2.17) $\beta_m \beta_l (p.T^i_{jk}) = a_{lm}\ (p.T^i_{jk})$.

(2.18) $\beta_m \beta_l (p.T^i_{j}) = a_{lm}\ (p.T^i_{j})$.

(2.19) $\beta_m \beta_l (p\ .H_{jk}) = a_{lm} (p\ .H_{jk})$.

(2.20) $\beta_m \beta_l (p\ .H_{j}) = a_{lm} (p\ .H_{j})$.

(2.21) $\beta_m \beta_l (T^i_{jkh} - T^a_{jkh} \ell^i \ell_a)$
$= a_{lm} (T^i_{jkh} - T^a_{jkh} \ell^i \ell_a)$.

(2.22) $\beta_m \beta_l (T^i_{jk} - T^a_{jk} \ell^i \ell_a)$
$= a_{lm} (T^i_{jk} - T^a_{jk} \ell^i \ell_a)$.

(2.23) $\beta_m \beta_l (T^i_{j} - T^a_{j} \ell^i \ell_a)$
$= a_{lm} (T^i_{j} - T^a_{j} \ell^i \ell_a)$.

3. Pseudo T-Trirecurrent Space:

In this section, we introduce a Finsler space which the curvature tensor T^i_{jkh} is trirecurrent in sense of Berwald .

Definition 3.1. A Finsler space F_n for which the curvature tensor T^i_{jkh} satisfies

(3.1) $\beta_n \beta_m \beta_l T^{\ i}_{jkh} = c_{lmn} T^i_{jkh}$, $T^i_{jkh} \neq 0$,

where c_{lmn} recurrence covariant tensor field of third order, this space will be called pseudo T- trirecurrent space.

Differentiating (2.13) covariantly with respect to x^n in sense of Berwald, we get

(3.2) $\beta_n \beta_m \beta_l T^{\ i}_{jkh} = (\beta_n a_{lm}) T^{\ i}_{jkh}$
$+ a_{lm} \beta_n T^{\ i}_{jkh}$.

In view of (2.11), the above equation becomes

$\beta_n \beta_m \beta_l T^{\ i}_{jkh} = (\beta_n a_{lm}) T^{\ i}_{jkh} + a_{lm} \lambda_n T^{\ i}_{jkh}$

which can be written as

$\beta_n \beta_m \beta_l T^{\ i}_{jkh} = c_{lmn} T^i_{jkh}$, $T^{\ i}_{jkh} \neq 0$,

where $c_{lmn} = (\beta_n a_{lm}) + a_{lm} \lambda_n$.

Thus, we conclude

Theorem 3.1. Every pseudo T −birecurrent space which the recurrence vector field satisfies $(\beta_n a_{lm}) + a_{lm}\lambda_n \neq 0$, is a pseudo T – trirecurrent space.

Transvecting (3.1) by y^j, using (2.5a) and (1.1a), we get

(3.3) $\beta_n\beta_m\beta_l T^{\ i}_{kh} = c_{lmn}T^i_{kh}$.

Transvecting (3.3) by y^k, using (2.5b) and (1.1a), we get

(3.4) $\beta_n\beta_m\beta_l T^{\ i}_{h} = c_{lmn}T^i_h$.

Thus, we conclude

Theorem 3.2. In pseudo T – trirecurrent space, The torsion tensor $T^{\ i}_{kh}$ and division tensor $T^{\ i}_{h}$ are trirecurrent.

Differentiating (1.3) covariantly with respect to x^n in sense of Berwald, we get

$$\beta_n\beta_m\beta_l H^{\ i}_{jkh} = (\beta_n a_{lm})H^{\ i}_{jkh} + a_{lm}\beta_n H^{\ i}_{jkh}.$$

Above equation can be written as

(3.5) $\beta_n\beta_m\beta_l H^{\ i}_{jkh} = c_{lmn}H^i_{jkh}\,,\ H^{\ i}_{jkh} \neq 0$,

Transvecting (3.5) by y^j, using (1.1a) and (2.7a), we get

(3.6) $\beta_n\beta_m\beta_l H^{\ i}_{kh} = c_{lmn}H^i_{kh}$.

Transvecting (3.6) by y^k, using (1.1a) and (2.7b), we get

(3.7) $\beta_n\beta_m\beta_l H^{\ i}_{h} = c_{lmn}H^i_h$.

Contracting i and h in (3.5) and using (2.7c), we get

(3.8) $\beta_n\beta_m\beta_l H_{jk} = c_{lmn}H_{jk}$

Transvecting (3.8) by y^j, using (1.1a) and (2.7d), we get

(3.9) $\beta_n\beta_m\beta_l H_k = c_{lmn}H_k$

Contracting i and h in (3.7) and using (2.7g), we get

(3.10) $\beta_n\beta_m\beta_l H = c_{lmn}H$

Thus, we conclude

Theorem 3.3. In flat pseudo T-trirecurrent space, the curvature tensor H^i_{jkh}, torsion tensor H^i_{kh}, division tensor H^i_h, Ricci tensor H_{jk}, curvature vector H_k and curvature scalar H are trirecurrent .

Differentiating (2.6a) covariantly with respect to x^l in sense of Berwald, we get

(3.11) $\beta_l T^i_{jkh} = \beta_l(W^i_{jkh} - H^i_{jkh})$

Differentiating (3.11) covariantly twice with respect to x^m and x^n , respectively, in sense of Berwald, we get

(3.12) $\beta_n\beta_m\beta_l T^i_{jkh} = \beta_n\beta_m\beta_l(W^i_{jkh} - H^i_{jkh})$,

Using (3.1), (3.5) and(2.6a) in (3.12), we get

(3.13) $\beta_n\beta_m\beta_l\, W^i_{jkh} = c_{lmn}W^i_{jkh}$

Transvecting (3.13) by y^j, using (2.8a) and (1.1a), we get

(3.14) $\beta_n\beta_m\beta_l\, W^i_{kh} = c_{lmn}\, W^i_{kh}$

Transvecting (3.14) by y^k, using (2.8b) and (1.1a), we get

(3.15) $\beta_n\beta_m\beta_l\, W^i_h = c_{lmn}\, W^i_h$

Thus, we conclude

Theorem3.4. In pseudo T-trirecurrent space, the projective curvature tensor W^i_{jkh}, the torsion tensor W^i_{kh} and the division tensor W^i_h are trirecurrent.

Using (3.1) and (2.6a) in (3.12) , we get

(3.16) $\beta_n\beta_m\beta_l(W^i_{jkh} - H^i_{jkh})$
$= c_{lmn}(W^i_{jkh} - H^i_{jkh})$.

Transvecting (3.16) by y^j, using (2.7a), (2.8a) and (1.1a), we get

(3.17) $\beta_n\beta_m\beta_l(W^i_{kh} - H^i_{kh})$
$= c_{lmn}(W^i_{kh} - H^i_{kh})$

Transvecting (3.17) by y^k, using (2.7b), (2.8b) and (1.1a), we get

(3.18) $\beta_n\beta_m\beta_l(W_h^i - H_h^i) = c_{lmn}(W_h^i - H_h^i)$,

Thus, we conclude

Theorem3.5. In pseudo T-trirecurrent space, The tensors $(W_{jkh}^i - H_{jkh}^i)$, $(W_{kh}^i - H_{kh}^i)$ and $(W_h^i - H_h^i)$ are trirecurrent.

4. Projection on Indicatrix with Respect to Berwald's Connection

Saleem and Abdallah introduced the projection on indicatrix for the tensors which be birecurrent [13]. In this section, we studied the projection on indicatrix for the tensors which be trirecurrent.

Let us consider a Finsler space F_n for which the curvature tensor T_{jkh}^i is trirecurrent in sense of Berwald, i.e. characterized by (3.1).

Now, in view of (1.2), the curvature tensor T_{jkh}^i on indicatrix is given by

(4.1) $p.T_{jkh}^i = T_{bcd}^a h_a^i h_j^b h_k^c h_h^d$.

Taking covariant derivative of (4.1) with respect to x^l , x^m and x^n in sense of Berwald and using the fact that $\beta_l h_j^i = 0$, we get

(4.2) $\beta_n\beta_m\beta_l(p\, T_{jkh}^i)$
$= \beta_n\beta_m\beta_l T_{bcd}^a h_a^i h_j^b h_k^c h_h^d$.

Using (3.1) in (4.2), we get

(4.3) $\beta_n\beta_m\beta_l(p\, T_{jkh}^i) = c_{lmn} T_{bcd}^a h_a^i h_j^b h_k^c h_h^d$.

In view of (1.2) and by using the fact that $\beta_l h_j^i = 0$, equation (4.3) can be written as

$\beta_n\beta_m\beta_l(p\, T_{jkh}^i) = c_{lmn}(p\,.T_{jkh}^i)$.

This shows that $p\,.T_{jkh}^i$ is trirecurrent.

Thus, we conclude

Theorem 4.1. The projection of the curvature tensor T_{jkh}^i on indicatrix in pseudo T-trirecurrent space is trirecurrent in sense of Berwald.

Taking covariant derivative of (2.17) with respect to x^n in sense of Berwald, we get

(4.4) $\beta_n\beta_m\beta_l(p\, T_{kh}^i) = \beta_n a_{lm}(p\,.T_{kh}^i)$.

Using (3.3) and using the fact that $\beta_l h_j^i = 0$ in (4.4) and in view of (1.2), we get

(4.5) $\beta_n\beta_m\beta_l(p\, T_{kh}^i) = c_{lmn}(p\,.T_{kh}^i)$.

Theorem 4.2. The projection of the torsion tensor T_{kh}^i on indicatrix in pseudo T – trirecurrent space is trirecurrent in sense of Berwald.

Taking covariant derivative of (2.18) with respect to x^n in sense of Berwald, we get

(4.6) $\beta_n\beta_m\beta_l(p\, T_h^i) = \beta_n a_{lm}(p\,.T_h^i)$.

Using (3.4) and the fact that $\beta_l h_j^i = 0$ in (4.6) and in view of (1.2), we get

(4.7) $\beta_n\beta_m\beta_l(p\, T_h^i) = c_{lmn}(p\,.T_h^i)$.

Theorem 4.3. The projection of the division tensor T_h^i on indicatrix in pseudo T – trirecurrent space is trirecurrent in sense of Berwald.

Taking covariant derivative of (2.21) with respect to x^n in sense of Berwald and using (2.11), we get

$$\beta_n\beta_m\beta_l(T_{jkh}^i - T_{jkh}^a \ell^i \ell_a)$$
$$= (\beta_n a_{lm} + \lambda_n a_{lm})(T_{jkh}^i - T_{jkh}^a \ell^i \ell_a).$$

Above equation we can written as

(4.8) $\beta_n\beta_m\beta_l(T_{jkh}^i - T_{jkh}^a \ell^i \ell_a)$

$= c_{lmn}(T^i_{jkh} - T^a_{jkh}\ell^i\ell_a)$.

Thus, we conclude

corollary 4.1. In pseudo T-trirecurrent space, the projection of the tensor T^i_{jkh} on indicatrix is trirecurrent if and only if $T^a_{jkh}\ell^i\ell_a$ is trirecurrent.

Taking covariant derivative of (2.22) with respect to x^n in sense of Berwald , we get

$$\beta_n\beta_m\beta_l(T^i_{jk} - T^a_{jk}\ell^i\ell_a)$$
$$= (\beta_n a_{lm} + \lambda_n a_{lm})(T^i_{jk} - T^a_{jk}\ell^i\ell_a).$$

Above equation can be written as

(4.9) $\beta_n\beta_m\beta_l(T^i_{jk} - T^a_{jk}\ell^i\ell_a)$

$= c_{lmn}(T^i_{jk} - T^a_{jk}\ell^i\ell_a)$.

Thus ,we conclude

corollary 4.2. In pseudo T-trirecurrent space, the projection of the torsion tensor T^i_{jk} on indicatrix is trirecurrent if and only if $T^a_{jk}\ell^i\ell_a$ is trirecurrent.

Taking covariant derivative of (2.23) with respect to x^n in sense of Berwald , we get

$$\beta_n\beta_m\beta_l(T^i_j - T^a_j\ell^i\ell_a)$$
$$= (\beta_n a_{lm} + \lambda_n a_{lm})(T^i_j - T^a_j\ell^i\ell_a).$$

Above equation can be written as

(4.10) $\beta_n\beta_m\beta_l(T^i_j - T^a_j\ell^i\ell_a)$

$= c_{lmn}(T^i_j - T^a_j\ell^i\ell_a)$.

Thus ,we conclude

corollary 4.3. in pseudo T – trirecurrent space, the projection of the division tensor T^i_j on indicatrix is trirecurrent if and only if $T^a_j\ell^i\ell_a$ is trirecurrent.

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فضاء فنسلر الزائف -T ثلاثي المعاودة وفق أسلوب بيرولد

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الملخص: في هذه الورقة البحثية قدمنا فضاء فنسلر الزائف- T ثلاثي المعاودة وفق أسلوب بيرولد وتوصلنا للعديد من النظريات في هذا الفضاء ثم قدمنا اسقاطات عدد من الموترات واثبتنا أنها تحقق الخاصية الثلاثية المعاودة في الفضاء الزائف – T ثلاثي المعاودة.

الكلمات المفتاحية: الفضاء الزائف-T ثلاثي المعاودة، المستوى الزائف-T ثلاثي المعاودة، الاسقاطات