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On Generalized R^h-Fourrecurrent Finsler Space

Amani M. A. Hanbala
Dept. of Maths., Community college -Aden

Wafa`a H. A. Hadi
Dept. of Maths., Community college-Aden

Abstract: In this paper, we introduce a Finsler space which Cartan's third curvature tensor R^i_{jkh} satisfies the generalized four recurrent property in sense of Cartan's, this space characterized by the following condition

$$R^i_{jkh|\ell|m|n|s} = u_{\ell mns} R^i_{jkh} + v_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk}), \quad R^i_{jkh} \neq 0,$$

where $|\ell|m|n|s$ is $h-$ covariant derivative of fourth order (Cartan's second kind covariant differential operator), with respect x^ℓ , x^m , x^n and x^s , respectively, where $u_{\ell mns}$ and $v_{\ell mns}$ are non-zero covariant tensor fields of fourth order called recurrence tensor fields, is introduced, such space is called as a generalized R^h- fourrecurrent Finsler space and we denote by $GR^h - FR - F_n$ and we obtained some generalized fourrecurrent in this space.

Keywords: Finsler space; Generalized R^h- fourrecurrent Finsler space; Ricci tensor.

1. Introduction: Finsler spaces have different connection, because this recurrence of different curvature tensor have been studied by various mathematicians. H.S. Ruse [11] considered a three dimensional Riemannian space having the recurrent of curvature tensor and he called such space as Riemannian space of recurrent curvature. This idea was extended to n-dimensional Riemannian and non- Riemannian space by A.G. Walker [6], Y.C. Worg and K. Yano [21] and others. The generalized curvature tensor in recurrent Finsler space used the sense of Berwald and Cartan curvature tensor discussed by AL-Qashbari and others ([1], [2] , [3] , [4] , [5] , [7] and [19]). The generalized birecurrent, trirecurrent Finsler space and higher order recurrent are studied in ([8], [9] , [11] , [12] , [14] and [15]). The recurrent of n-dimensional space was extended to Finsler space ([10] and [20]) for the first time.

Due to different connections of Finsler space, the recurrent of Cartan`s third curvature tensor $R^i_{jkh}$ have been discussed by, R. Verma [17] , birecurrent of Cartan`s third curvature tensor R^i_{jkh} have been discussed by S. Dikshit [18] and the generalized birecurrent of Cartan`s third curvature tensor R^i_{jkh} have been discussed by F. Y. A. Qasem [10] .P. N. Pandey, S. Saxena and A. Goswani [16] introduced a generalized H-recurrent Finsler space.

Let F_n be An n-dimensional Finsler space equipped with the metric function a $F(x, y)$ satisfying the request conditions [11] .

The vectors y_i , $y^i$and the metric tensor g_{ij} satisfies the following relations

(1.1) a) $y_i\, y^i = F^2$,

b) $g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i$, c) $y_{i|k} = 0$

d) $y^i_{\ |k} = 0$, e) $g_{ij|k} = 0$

f) $g^{ij}_{|k} = 0$,

g) $g_{ij} g^{jk} = \delta^k_i = \begin{cases} 1 & if \quad i = k \ , \\ 0 & if \quad i \neq k \ . \end{cases}$

The two processes of covariant differentiation, defined above commute with the partial

(1.2) $a)\ \dot{\partial}_j (X^i_{\ |k}) - (\dot{\partial}_j X^i)_{|k}$

$= X^r (\dot{\partial}_j \Gamma^{*i}_{rk}) - (\dot{\partial}_r X^i)\, P^r_{jk}$,

b) $g_{ir}\, P^i_{kh} = P_{k.rh}$.

The vector y_i , metric g_{ij} and δ^i_k also satisfy the following relations

(1.3) a) $\delta^i_k\, y^k = y^i$, b) $\delta^i_k\, y_i = y_k$

(1.4) a) $\delta^i_k\, g_{ji} = g_{jk}$, b) $g_{jh}\, y^j = y_h$.

The associate curvature tensor R_{ijkh} of the curvature tensor R^i_{jkh} is given by

(1.5) a) $R_{ijkh} = g_{rj}\, R^r_{ikh}$ and

b) $R_{jrkh}\, g^{ir} = R^i_{jkh}$

The R –Ricci tensor R_{jk} of the curvature tensor R^i_{jkh} , the tensor R^r_h , the curvature scalar R and the deviation tensor R_j is given by

(1.6) a) $R^i_{jki} = R_{jk}$, b) $R_{jk}\, g^{jk} = R$

c) $R_{jk}\, y^k = R_j$ and d) $R^r_{ikh} g^{ik} = R^r_h$.

Cartan's third curvature tensor R^i_{jkh} and the Berwald curvature tensor H^i_{jkh} , satisfies the relation

(1.7) a) $R^i_{jkh} y^j = H^i_{kh}$, b) $H^i_{jkh} = \dot{\partial}_j H^i_{kh}$

and c) $R^i_{jkh} = -R^i_{jhk}$.

They are also related by [14]

(1.8) a) $g_{ip}\, H^i_{jkh} = H_{jpkh}$ and

b) $H^i_{jk} = \dot{\partial}_j H^i_k$.

The torsion tensor H^i_{kh} satisfies

(1.9) $H^i_{kh}\, y^k = H^i_h = -H^i_{hk}\, y^k$,

(1.10) $H_{jk} = H^i_{jki}$,

(1.11) $H_k = H^i_{ki}$,

(1.12) $g_{ip}\, H^i_{jk} = H_{jpk}$ and

(1.13) $H = \frac{1}{n-1}\, H^i_i$.

The hv –curvature tensor P^i_{jkh} is positively homogeneous of degree zero in y^i and satisfies the relation

(1.14) $P^i_{jkh} y^j = \Gamma^{*i}_{jhk} y^j = P^i_{kh} = C^i_{kh|r} y^r$.

Definition 1.1: For Riemannian space v_n, the projection curvature tensor P^i_{jkh} (Cartan's second curvature tensor) is defined as [22]

(1.15) $P^i_{jkh} = R^i_{jkh} - \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)$

2. On Necessary and Sufficient Condition of Generalized R^h-Fourrecurrent Finsler Space

Let us consider a Finsler space F_n in which Cartan's third curvature tensor R^i_{jkh} satisfied the following generalized recurrence condition ([4])

(2.1) $R^i_{jkh|\ell} = \lambda_\ell R^i_{jkh} + \mu_\ell(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$, $R^i_{jkh} \neq 0$, where $|\ell$ is h-covariant derivative of first order (Cartan's second kind covariant differential operator), with respect to x^ℓ and λ_ℓ and μ_ℓ are non-zero covariant vector fields and called the recurrence vector fields. Such space called it as a generalized R^h- recurrent Finsler space.

Consider a Finsler space F_n, whose Cartan's third curvature tensor R^i_{jkh} satisfied the following generalized birecurrence condition

(2.2) $R^i_{jkh|\ell|m} = a_{\ell m} R^i_{jkh} + b_{\ell m}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$, $R^i_{jkh} \neq 0$,

where $|\ell|m$ is h-covariant derivative of second order with respect to x^ℓ and x^m , respectively, where $a_{\ell m} = \lambda_{\ell|m} + \lambda_\ell\, \lambda_m$ and $b_{\ell m} = \lambda_\ell \mu_m + \mu_{\ell|m}$ are non-zero covariant tensor fields of second order and called recurrence tensor fields. Such space called it as a generalized R^h-birecurrent Finsler space.

Taking h -covariant derivative of (2.2), with respect to x^n and using (1.1d), (1.1e) and (2.1), we get

(2.3) $R^i_{jkh|\ell|m|n} = c_{\ell mn} R^i_{jkh} + d_{\ell mn}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$, $R^i_{jkh} \neq 0$,

where $|\ell|m|n$ is h – covariant derivative of third order with respect to x^ℓ, x^m and x^n , respectively, where $c_{\ell mn} = a_{\ell m|n} + a_{\ell m}\lambda_n$ and $b_{\ell mn} = a_{\ell m}\mu_n + b_{\ell m|n}$ are non-zero covariant tensor fields of third order and called recurrence tensor fields. Such space called it as a generalized R^h- trirecurrent Finsler space.

Taking h-covariant derivative of (2.3), with respect to x^s and using (1.1d), (1.1e) and (2.1), we get

(2.4) $R^i_{jkh|\ell|m|n|s} = u_{\ell mns} R^i_{jkh} + v_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$,

where $|\ell|m|n|s$ is h – covariant derivative of four order with respect to x^ℓ, x^m , x^n and x^s , respectively, where $u_{\ell mns} = c_{\ell mn|s} + c_{\ell mn}\,\lambda_n$ and $v_{\ell mns} = c_{\ell mn}\mu_s + d_{\ell mn|s}$ are non-zero covariant tensor fields of fourth order and called recurrence tensor fields. Such space called it as a generalized R^h-fourrecurrent Finsler space.

Definition 2.1. If Cartan's third curvature tensor R^i_{jkh} of a Finsler space satisfying the condition (2.4), where $u_{\ell mns}$ and $v_{\ell mns}$ are non-zero covariant tensor fields of fourth order , the space and the tensor will be called generalized R^h-fourrecurrent Finsler space, we shall denote such space briefly by GR^h-FR-F_n .

However, if we start from condition (2.4), we cannot obtain the condition (2.1), we may conclude

Theorem 2.1. Every generalized R^h – recurrent Finsler space is generalized R^h –

fourrecurrent Finsler space, but the converse need not be true.

Transvecting (2.4) by the metric tensor g_{ir} , using (1.1e), (1.5a) and (1.4a) ,we get

(2.5) $R_{jrkh|\ell|m|n|s} = u_{\ell mns} R_{jrkh}$
$+ v_{\ell mns}(g_{kr}\, g_{jh} - g_{hr}\, g_{jk})$.

Conversely, the transvection of the condition (2.5) by g^{ir}, by using (1.1f), (1.5b) and (1.1g), yield the condition (2.4).

Thus, we may conclude

Theorem 2.2. In GR^h-FR-F_n , the h – covariant derivative of fourth order for the associate curvature tensor R_{jrkh} of Cartan's third curvature tensor R^i_{jkh} is given by (2.5).

Transvecting the condition (2.4) by y^j , using (1.1d), (1.4b) and (1.7a) we get

(2.6) $H^i_{kh|\ell|m|n|s} = u_{\ell mns} H^i_{kh}$
$+ v_{\ell mns}(\delta^i_k y_h - \delta^i_h y_k)$.

Further transvecting (2.6) by y^k , using (1.1d), (1.1a), (1.3a) and (1.9), we get

(2.7) $H^i_{h|\ell|m|n|s} = u_{\ell mns} H^i_h$
$+ v_{\ell mns}(y^i y_h - \delta^i_h F^2)$.

Thus, we may conclude

Theorem 2.3. In GR^h-FR-F_n , the h – covariant derivative of fourth order for the h(v)-torsion tenser H^i_{kh} and the deviation tensor H^i_h given by (2.6) and (2.7), respectively.

Contracting the indices i and h in equations (2.4), (2.6) and (2.7) , using (1.6a), (1.11), (1.13) and (1.3b),(1.1g) and (1.4a) we get

(2.8) $R_{jk|\ell|m|n|s} = u_{\ell mns} R_{jk}$
$+ (1-n) v_{\ell mns}\, g_{jk}$.

(2.9) $H_{k|\ell|m|n|s} = u_{\ell mns} H_k$
$+ (1-n) v_{\ell mns}\, y_k$.

(2.10) $H_{|\ell|m|n|s} = u_{\ell mns} H - v_{\ell mns} F^2$.

Transvecting (2.4) and (2.8) by g^{jk} , using (1.1f), (1.6d) and (1.6b), we get

(2.11) $R^i_{h|\ell|m|n|s} = u_{\ell mns}\, R^i_h$
$+ v_{\ell mns}(y^i y_h - \delta^i_h)$

(2.12) $R_{|\ell|m|n|s} = u_{\ell mns}\, R$
$+ (1-n)\, v_{\ell mns}$.

Thus, we conclude

Theorem 2.4. The Ricci tensor R_{jk}, the curvature vector H_k ,the scalar curvature H the deviation tensor R^i_h and the scalar curvature tensor R are behave as fourrecurrent in GR^h-FR-F_n .

Differentiating (2.6) partially with respect to y^j , using (1.7b) and (1.1b), we get

(2.13) $\dot\partial_j \left(H^i_{kh|\ell|m|n|s}\right) = (\dot\partial_j u_{\ell mns}) H^i_{kh}$
$+ u_{\ell mns} H^i_{jkh} + (\dot\partial_j v_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k)$
$+ v_{\ell mns}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$

Using commutation formula exhibited by (1.2a) for $(H^i_{kh|\ell|m|n})$ in (2.13), we get

(2.14) $\left\{\dot\partial_j \left(H^i_{kh|\ell|m|n}\right)\right\}_{|s} + H^r_{kh|\ell|m|n}(\dot\partial_j \Gamma^{*i}_{rs})$
$-H^i_{rh|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{ks}) - H^i_{rk|\ell|m|n}(\dot\partial_j \Gamma^{*r}_{hs})$
$- H^i_{kh|r|m|n}(\dot\partial_j \Gamma^{*r}_{\ell s}) - H^i_{kh|\ell|r|n}(\dot\partial_j \Gamma^{*r}_{ms})$
$-H^i_{kh|\ell|m|r}(\dot\partial_j \Gamma^{*r}_{ns}) - \dot\partial_r \left(H^i_{kh|\ell|m|n}\right) P^r_{js}$
$= (\dot\partial_j u_{\ell mns}) H^i_{kh} + u_{\ell mns} H^i_{jkh}$
$+ (\dot\partial_j v_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k)$
$+ v_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Again applying the commutation formula exhibited by (1.2a) for $(H^i_{kh|\ell|m})$ in (2.14) and using (1.7b), we get

(2.15) $\left\{\dot{\partial}_j\left(H^i_{kh|\ell|m}\right)\right\}_{|n|s}+\left[H^r_{kh|\ell|m}\left(\dot{\partial}_j\Gamma^{*i}_{rn}\right)\right.$
$-H^i_{rh|\ell|m}\left(\dot{\partial}_j\Gamma^{*r}_{kn}\right)-H^i_{kr|\ell|m}\left(\dot{\partial}_j\Gamma^{*r}_{hn}\right)$
$-H^i_{kh|r|m}\left(\dot{\partial}_j\Gamma^{*r}_{\ell n}\right)-H^i_{kh|\ell|r}\left(\dot{\partial}_j\Gamma^{*r}_{mn}\right)$
$\left.-\dot{\partial}_r\left(H^i_{kh|\ell|m}\right)P^r_{jn}\right]_{|s}+H^r_{kh|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*i}_{rs}\right)$
$-H^i_{rh|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{ks}\right)-H^i_{kr|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{hs}\right)$
$-H^i_{kh|r|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{\ell s}\right)-H^i_{kh|\ell|r|n}\left(\dot{\partial}_j\Gamma^{*r}_{ms}\right)$
$-H^i_{kh|\ell|m|r}\left(\dot{\partial}_j\Gamma^{*r}_{ns}\right)-\dot{\partial}_r\left(H^i_{kh|\ell|m|n}\right)P^r_{js}$
$=\left(\dot{\partial}_j u_{\ell mns}\right)H^i_{kh}+u_{\ell mns}H^i_{jkh}$
$+\left(\dot{\partial}_j v_{\ell mns}\right)\left(\delta^i_k y_h-\delta^i_h y_k\right)$
$+v_{\ell mns}\left(\delta^i_k g_{jh}-\delta^i_h g_{jk}\right).$

Again applying the commutation formula exhibited by (1.2a) for $(H^i_{kh|\ell})$ in (2.15) and using (1.7b), we get

(2.16) $\left\{\dot{\partial}_j\left(H^i_{kh|\ell}\right)\right\}_{|m|n|s}+\left[H^r_{kh|\ell}\left(\dot{\partial}_j\Gamma^{*i}_{rm}\right)\right.$
$-H^i_{rh|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{km}\right)-H^i_{kr|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{hm}\right)$
$\left.-H^i_{kh|r}\left(\dot{\partial}_j\Gamma^{*r}_{\ell m}\right)-\dot{\partial}_r\left(H^i_{kh|\ell}\right)P^r_{jm}\right]_{|n|s}$
$+\left[H^r_{kh|\ell|m}\left(\dot{\partial}_j\Gamma^{*i}_{rn}\right)-H^i_{rh|\ell|m}\left(\dot{\partial}_j\Gamma^{*r}_{kn}\right)\right.$
$-H^i_{kr|\ell|m}\left(\dot{\partial}_j\Gamma^{*r}_{hn}\right)-H^i_{kh|r|m}\left(\dot{\partial}_j\Gamma^{*r}_{\ell n}\right)$
$\left.-H^i_{kh|\ell|r}\left(\dot{\partial}_j\Gamma^{*r}_{mn}\right)-\dot{\partial}_r\left(H^i_{kh|\ell|m}\right)P^r_{jn}\right]_{s}$
$+H^r_{kh|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*i}_{rs}\right)-H^i_{rh|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{ks}\right)$
$-H^i_{kr|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{hs}\right)-H^i_{kh|r|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{\ell s}\right)$
$-H^i_{kh|\ell|r|n}\left(\dot{\partial}_j\Gamma^{*r}_{ms}\right)-H^i_{kh|\ell|m|r}\left(\dot{\partial}_j\Gamma^{*r}_{ns}\right)$
$-\dot{\partial}_r\left(H^i_{kh|\ell|m|n}\right)P^r_{js}=\left(\dot{\partial}_j u_{\ell mns}\right)H^i_{kh}$
$+u_{\ell mns}H^i_{jkh}+\left(\dot{\partial}_j v_{\ell mns}\right)\left(\delta^i_k y_h-\delta^i_h y_k\right)$
$+v_{\ell mns}\left(\delta^i_k g_{jh}-\delta^i_h g_{jk}\right).$

Further, applying the commutation formula exhibited by (1.2a) for (H^i_{kh}) in (2.16) and using (1.7b), we get

(2.17) $\left\{\dot{\partial}_j\left(H^i_{kh}\right)\right\}_{|\ell|m|n|s}+\left[H^r_{kh}\left(\dot{\partial}_j\Gamma^{*i}_{r\ell}\right)-\right.$
$\left.H^i_{rh|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{k\ell}\right)-H^i_{kr|\ell}-\dot{\partial}_r\left(H^i_{kh}\right)P^r_{j\ell}\right]_{|m|n|s}+$
$\left[H^r_{kh|\ell}\left(\dot{\partial}_j\Gamma^{*i}_{rm}\right)+H^i_{rh|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{km}\right)-\right.$
$H^i_{kr|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{hm}\right)-H^i_{kh|r}\left(\dot{\partial}_j\Gamma^{*r}_{\ell m}\right)-$
$\left.\dot{\partial}_r\left(H^i_{kh|\ell}\right)P^r_{jm}\right]_{|n|s}+\left[H^r_{kh|\ell|m}\left(\dot{\partial}_j\Gamma^{*i}_{rn}\right)-\right.$
$H^i_{rh|\ell|m}\left(\dot{\partial}_j\Gamma^{*r}_{kn}\right)-H^i_{kr|\ell|m}\left(\dot{\partial}_j\Gamma^{*r}_{hn}\right)-$
$H^i_{kh|r|m}\left(\dot{\partial}_j\Gamma^{*r}_{\ell n}\right)-H^i_{kh|\ell|r}\left(\dot{\partial}_j\Gamma^{*r}_{mn}\right)$
$\left.-\dot{\partial}_r\left(H^i_{kh|\ell|m}\right)P^r_{jn}\right]_{|s}+H^r_{kh|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*i}_{rs}\right)$
$-H^i_{rh|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{ks}\right)-H^i_{kr|\ell|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{hs}\right)$
$-H^i_{kh|r|m|n}\left(\dot{\partial}_j\Gamma^{*r}_{\ell s}\right)-H^i_{kh|\ell|r|n}\left(\dot{\partial}_j\Gamma^{*r}_{ms}\right)$
$-H^i_{kh|\ell|m|r}\left(\dot{\partial}_j\Gamma^{*r}_{ns}\right)-\dot{\partial}_r\left(H^i_{kh|\ell|m|n}\right)P^r_{js}$
$=\left(\dot{\partial}_j u_{\ell mns}\right)H^i_{kh}+u_{\ell mns}H^i_{jkh}$
$+\left(\dot{\partial}_j v_{\ell mns}\right)\left(\delta^i_k y_h-\delta^i_h y_k\right)+$
$v_{\ell mns}\left(\delta^i_k g_{jh}-\delta^i_h g_{jk}\right).$

Using (1.7b) in (2.17), we get

(2.18) $H^i_{jkh|\ell|m|n|s}+\left[H^r_{kh}\left(\dot{\partial}_j\Gamma^{*i}_{r\ell}\right)\right.$
$-H^i_{rh}\left(\dot{\partial}_j\Gamma^{*r}_{k\ell}\right)-H^i_{kr}\left(\dot{\partial}_j\Gamma^{*r}_{h\ell}\right)$
$\left.-\dot{\partial}_r\left(H^i_{kh}\right)P^r_{j\ell}\right]_{|m|n|s}+\left[H^r_{kh|\ell}\left(\dot{\partial}_j\Gamma^{*i}_{rm}\right)\right.$
$-H^i_{rh|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{km}\right)-H^i_{kr|\ell}\left(\dot{\partial}_j\Gamma^{*r}_{hm}\right)$
$\left.-H^i_{kh|r}\left(\dot{\partial}_j\Gamma^{*r}_{\ell m}\right)-\dot{\partial}_r\left(H^i_{kh|\ell}\right)P^r_{jm}\right]_{|n|s}$
$+\left[H^r_{kh|\ell|m}\left(\dot{\partial}_j\Gamma^{*i}_{rn}\right)-H^i_{rh|\ell|m}\left(\dot{\partial}_j\Gamma^{*r}_{kn}\right)\right.$
$-H^i_{kr|\ell|m}\left(\dot{\partial}_j\Gamma^{*r}_{hn}\right)-H^i_{kh|r|m}\left(\dot{\partial}_j\Gamma^{*r}_{\ell n}\right)$
$\left.-H^i_{kh|\ell|r}\left(\dot{\partial}_j\Gamma^{*r}_{mn}\right)-\dot{\partial}_r\left(H^i_{kh|\ell|m}\right)P^r_{jn}\right]_{|s}$

$+H^r_{kh|\ell|m|n}(\dot\partial_j\Gamma^{*i}_{rs}) - H^i_{rh|\ell|m|n}(\dot\partial_j\Gamma^{*r}_{ks})$
$-H^i_{kr|\ell|m|n}(\dot\partial_j\Gamma^{*r}_{hs}) - H^i_{kh|r|m|n}(\dot\partial_j\Gamma^{*r}_{\ell s})$
$-H^i_{kh|\ell|r|n}(\dot\partial_j\Gamma^{*r}_{ms}) - H^i_{kh|\ell|m|r}(\dot\partial_j\Gamma^{*r}_{ns}) -$
$\dot\partial_r\left(H^i_{kh|\ell|m|n}\right)P^r_{js} = (\dot\partial_j u_{\ell mns})H^i_{kh}$
$+u_{\ell mns}H^i_{jkh} + (\dot\partial_j v_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k)$
$+v_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$

This shows that

(2.19) $H^i_{jkh|\ell|m|n|s} = u_{\ell mns}H^i_{jkh}$
$+v_{\ell mns}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$

if and only if

(2.20) $\left[H^r_{kh}(\dot\partial_j\Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot\partial_j\Gamma^{*r}_{k\ell}) - H^i_{kr}(\dot\partial_j\Gamma^{*r}_{h\ell}) - \dot\partial_r(H^i_{kh})P^r_{j\ell}\right]_{|m|n|s}$
$+\left[H^r_{kh|\ell}(\dot\partial_j\Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j\Gamma^{*r}_{km})\right.$
$-H^i_{kr|\ell}(\dot\partial_j\Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j\Gamma^{*r}_{\ell m})$
$\left.-\dot\partial_r\left(H^i_{kh|\ell}\right)P^r_{jm}\right]_{|n|s} + \left[H^r_{kh|\ell|m}(\dot\partial_j\Gamma^{*i}_{rn})\right.$
$-H^i_{rh|\ell|m}(\dot\partial_j\Gamma^{*r}_{kn}) - H^i_{kr|\ell|m}(\dot\partial_j\Gamma^{*r}_{hn})$
$-H^i_{kh|r|m}(\dot\partial_j\Gamma^{*r}_{\ell n}) - H^i_{kh|\ell|r}(\dot\partial_j\Gamma^{*r}_{mn})$
$\left.-\dot\partial_r\left(H^i_{kh|\ell|m}\right)P^r_{jn}\right]_{|s} + H^r_{kh|\ell|m|n}(\dot\partial_j\Gamma^{*i}_{rs})$
$-H^i_{rh|\ell|m|n}(\dot\partial_j\Gamma^{*r}_{ks}) - H^i_{kr|\ell|m|n}(\dot\partial_j\Gamma^{*r}_{hs})$
$-H^i_{kh|r|m|n}(\dot\partial_j\Gamma^{*r}_{\ell s}) - H^i_{kh|\ell|r|n}(\dot\partial_j\Gamma^{*r}_{ms})$
$-H^i_{kh|\ell|m|r}(\dot\partial_j\Gamma^{*r}_{ns}) - \dot\partial_r\left(H^i_{kh|\ell|m|n}\right)P^r_{js}$
$-(\dot\partial_j u_{\ell mns})H^i_{kh}$
$-(\dot\partial_j v_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) = 0.$

Thus, we may conclude

Theorem 2.5. In GR^h-FR-F_n , Berwald curvature tensor H^i_{jkh} is generalized fourrecurrent tensor if and only if (2.20) hold good.

Transvecting (2.18) by g_{ip}, using (1.1e), (1.8a), (1.12) and (1.4a), we get

(2.21) $H_{jpkh|\ell|m|n|s} + \left[g_{ip}H^r_{kh}(\dot\partial_j\Gamma^{*i}_{r\ell}) - H_{rph}(\dot\partial_j\Gamma^{*r}_{k\ell}) - H_{kpr}(\dot\partial_j\Gamma^{*r}_{h\ell}) - g_{ip}(\dot\partial_r H^i_{kh})P^r_{j\ell}\right]_{|m|n|s}$
$+\left[g_{ip}H^r_{kh|\ell}(\dot\partial_j\Gamma^{*i}_{rm}) - H_{rph|\ell}(\dot\partial_j\Gamma^{*r}_{km})\right.$
$-H_{kpr|\ell}(\dot\partial_j\Gamma^{*r}_{hm})$
$-H_{kph|r}(\dot\partial_j\Gamma^{*r}_{\ell m})$
$\left.-g_{ip}\dot\partial_r\left(H^i_{kh|\ell}\right)P^r_{jm}\right]_{|n|s}$
$+\left[g_{ip}H^r_{kh|\ell|m}(\dot\partial_j\Gamma^{*i}_{rn}) - H_{rph|\ell|m}(\dot\partial_j\Gamma^{*r}_{kn})\right.$
$-H_{kpr|\ell|m}(\dot\partial_j\Gamma^{*r}_{hn})$
$-H_{kph|r|m}(\dot\partial_j\Gamma^{*r}_{\ell n})$
$-H_{kph|\ell|r}(\dot\partial_j\Gamma^{*r}_{mn})$
$\left.-g_{ip}\dot\partial_r\left(H^i_{kh|\ell|m}\right)P^r_{jn}\right]_{|s}$
$+g_{ip}H^r_{kh|\ell|m|n}(\dot\partial_j\Gamma^{*i}_{rs}) - H_{rph|\ell|m|n}(\dot\partial_j\Gamma^{*r}_{ks})$
$-H_{kpr|\ell|m|n}(\dot\partial_j\Gamma^{*r}_{hs})$
$-H_{kph|r|m|n}(\dot\partial_j\Gamma^{*r}_{\ell s}) - H_{kph|\ell|r|n}(\dot\partial_j\Gamma^{*r}_{ms}) - H_{kph|\ell|m|r}(\dot\partial_j\Gamma^{*r}_{ns}) - g_{ip}\dot\partial_r\left(H^i_{kh|\ell|m|n}\right)P^r_{js}$
$= (\dot\partial_j u_{\ell mns})H_{kph} + u_{\ell mns}H_{jpkh}$
$+g_{ip}(\dot\partial_j v_{\ell mns})(\delta^i_k y_h - \delta^i_h y_k) + v_{\ell mns}(g_{kp}\,g_{jh} - g_{hp}\,g_{jk}).$

This shows that

(2.22) $H_{jpkh|\ell|m|n|s} = u_{\ell mns}H_{jpkh}$
$+v_{\ell mns}(g_{kp}\,g_{jh} - g_{hp}\,g_{jk}).$

if and only if

(2.23) $\left[g_{ip}H^r_{kh}(\dot\partial_j\Gamma^{*i}_{r\ell}) - H_{rph}(\dot\partial_j\Gamma^{*r}_{k\ell}) - H_{kpr}(\dot\partial_j\Gamma^{*r}_{h\ell}) - g_{ip}(\dot\partial_r H^i_{kh})P^r_{j\ell}\right]_{|m|n|s}$

$+\Big[g_{ip}\,H^{r}_{kh|\ell}\big(\dot{\partial}_j\Gamma^{*i}_{rm}\big) - H_{rph|\ell}\big(\dot{\partial}_j\Gamma^{*r}_{km}\big)$

$- H_{kpr|\ell}\big(\dot{\partial}_j\Gamma^{*r}_{hm}\big)$

$- H_{kph|r}\big(\dot{\partial}_j\Gamma^{*r}_{\ell m}\big)$

$- g_{ip}\dot{\partial}_r\left(H^{i}_{kh|\ell}\right)P^{r}_{jm}\Big]_{|n|s}$

$+\Big[g_{ip}\,H^{r}_{kh|\ell|m}\big(\dot{\partial}_j\Gamma^{*i}_{rn}\big) - H_{rph|\ell|m}\big(\dot{\partial}_j\Gamma^{*r}_{kn}\big)$

$- H_{kpr|\ell|m}\big(\dot{\partial}_j\Gamma^{*r}_{hn}\big)$

$- H_{kph|r|m}\big(\dot{\partial}_j\Gamma^{*r}_{\ell n}\big)$

$- H_{kph|\ell|r}\big(\dot{\partial}_j\Gamma^{*r}_{mn}\big)$

$-g_{ip}\,\dot{\partial}_r\left(H^{i}_{kh|\ell|m}\right)P^{r}_{jn}\Big]_{|s}$

$+ g_{ip}\,H^{r}_{kh|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*i}_{rs}\big)$

$- H_{rph|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{ks}\big)$

$- H_{kpr|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{hs}\big)$

$-H_{kph|r|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{\ell s}\big) - H_{kph|\ell|r|n}\big(\dot{\partial}_j\Gamma^{*r}_{ms}\big) -$
$H_{kph|\ell|m|r}\big(\dot{\partial}_j\Gamma^{*r}_{ns}\big) - g_{ip}\dot{\partial}_r\left(H^{i}_{kh|\ell|m|n}\right)P^{r}_{js}$
$-\big(\dot{\partial}_j u_{\ell mns}\big)H_{kph} + g_{ip}\big(\dot{\partial}_j v_{\ell mns}\big)\big(\delta^i_k y_h -$
$\delta^i_h y_k\big) = 0.$

Thus, we may conclude

Theorem 2.6. In GR^h-FR-F_n, the associative curvature tensor H_{jpkh} of Berwald curvature tensor H^i_{jkh} is generalized fourrecurrent tensor if and only if (2.23) hold good.

Contracting the i and h in (2.18), using (1.3b), (1.4a), (1.11) and (1.10), we get

(2.24) $H_{jk|\ell|m|n|s} + \big[H^{r}_{ki}\big(\dot{\partial}_j\Gamma^{*i}_{r\ell}\big) -$
$H_r\big(\dot{\partial}_j\Gamma^{*r}_{k\ell}\big) - H^{i}_{kr}\big(\dot{\partial}_j\Gamma^{*r}_{i\ell}\big) - \dot{\partial}_r(H_k)P^{r}_{j\ell}\big]_{|m|n|s}$
$+\Big[H^{r}_{ki|\ell}\big(\dot{\partial}_j\Gamma^{*i}_{rm}\big) + \big[-H_{r|\ell}\big(\dot{\partial}_j\Gamma^{*r}_{km}\big) -$
$H^{i}_{kr|\ell}\big(\dot{\partial}_j\Gamma^{*r}_{im}\big) - H_{k|r}\big(\dot{\partial}_j\Gamma^{*r}_{\ell m}\big) -$
$\dot{\partial}_r\big(H_{k|\ell}\big)P^{r}_{jm}\Big]_{|n|s} + \Big[H^{r}_{ki|\ell|m}\big(\dot{\partial}_j\Gamma^{*i}_{rn}\big) -$
$H_{r|\ell|m}\big(\dot{\partial}_j\Gamma^{*r}_{kn}\big) - H^{i}_{kr|\ell|m}\big(\dot{\partial}_j\Gamma^{*r}_{in}\big) -$
$H_{k|r|m}\big(\dot{\partial}_j\Gamma^{*r}_{\ell n}\big) - H_{k|\ell|r}\big(\dot{\partial}_j\Gamma^{*r}_{mn}\big)$
$-\dot{\partial}_r\big(H_{k|\ell|m}\big)P^{r}_{jn}\Big]_{|s} + H^{r}_{ki|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*i}_{rs}\big)$
$-H_{r|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{ks}\big) - H^{i}_{kr|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{is}\big)$
$- H_{k|r|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{\ell s}\big) - H_{k|\ell|r|n}\big(\dot{\partial}_j\Gamma^{*r}_{ms}\big)$
$-H_{k|\ell|m|r}\big(\dot{\partial}_j\Gamma^{*r}_{ns}\big) - \dot{\partial}_r\big(H_{k|\ell|m|n}\big)P^{r}_{js}$
$= \big(\dot{\partial}_j u_{\ell mns}\big)H_k + u_{\ell mns}H_{jk}$
$+ \big(\dot{\partial}_j v_{\ell mns}\big)(1-n)y_k + v_{\ell mns}(1-n)g_{jk}.$

This shows that

(2.25) $H_{jk|\ell|m|n|s} = u_{\ell mns}\,H_{jk}$
$+(1-n)v_{\ell mns}\,g_{jk}.$

if and only if

(2.26) $\big[H^{r}_{ki}\big(\dot{\partial}_j\Gamma^{*i}_{r\ell}\big) - H_r\big(\dot{\partial}_j\Gamma^{*r}_{k\ell}\big) -$
$H^{i}_{kr}\big(\dot{\partial}_j\Gamma^{*r}_{i\ell}\big) - \dot{\partial}_r(H_k)P^{r}_{j\ell}\big]_{|m|n|s}$

$\Big[H^{r}_{ki|\ell}\big(\dot{\partial}_j\Gamma^{*i}_{rm}\big) - H_{r|\ell}\big(\dot{\partial}_j\Gamma^{*r}_{km}\big)$
$-H^{i}_{kr|\ell}\big(\dot{\partial}_j\Gamma^{*r}_{im}\big) - H_{k|r}\big(\dot{\partial}_j\Gamma^{*r}_{\ell m}\big)$
$- \dot{\partial}_r\big(H_{k|\ell}\big)P^{r}_{jm}\Big]_{|n|s}$

$+\Big[H^{r}_{ki|\ell|m}\big(\dot{\partial}_j\Gamma^{*i}_{rn}\big) - H_{r|\ell|m}\big(\dot{\partial}_j\Gamma^{*r}_{kn}\big)$
$-H^{i}_{kr|\ell|m}\big(\dot{\partial}_j\Gamma^{*r}_{in}\big) - H_{k|r|m}\big(\dot{\partial}_j\Gamma^{*r}_{\ell n}\big)$
$-H_{k|\ell|r}\big(\dot{\partial}_j\Gamma^{*r}_{mn}\big) - \dot{\partial}_r\big(H_{k|\ell|m}\big)P^{r}_{jn}\Big]_{|s}$
$+H^{r}_{ki|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*i}_{rs}\big) - H_{r|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{ks}\big)$
$-H^{i}_{kr|\ell|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{is}\big) - H_{k|r|m|n}\big(\dot{\partial}_j\Gamma^{*r}_{\ell s}\big)$
$- H_{k|\ell|r|n}\big(\dot{\partial}_j\Gamma^{*r}_{ms}\big) - H_{k|\ell|m|r}\big(\dot{\partial}_j\Gamma^{*r}_{ns}\big)$
$-\dot{\partial}_r\big(H_{k|\ell|m|n}\big)P^{r}_{js} - \big(\dot{\partial}_j u_{\ell mns}\big)H_k$
$-\big(\dot{\partial}_j v_{\ell mns}\big)(1-n)y_k = 0.$

Thus , we have

Theorem 2.7. In GR^h-FR-F_n , Ricci curvature tensor H_{jk} is non-vanishing if and only if condition (2.26) hold good.

We can written (1.15) as

(2.27) $R^i_{jkh} = P^i_{jkh} + \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)$

Taking the $h-$ covariant derivative fourth for (2.27) with respect to x^ℓ, x^m, x^n and x^s, respectively, we get

(2.28) $R^i_{jkh|\ell|m|n|s} = \left(P^i_{jkh} + \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right)_{|\ell|m|n|s}$

Using condition (2.4) and (2.27) in (2.28), we get

$u_{\ell mns} R^i_{jkh} + v_{\ell ms}\left(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}\right) = \left(P^i_{jkh} + \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right)_{|\ell|m|n|s}$

(2.29) $\left(P^i_{jkh} + \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right)_{|\ell|m|n|s} = u_{\ell mns}\left(P^i_{jkh} + \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right) + v_{\ell ms}\left(\delta^i_k g_{jh} - \delta^i_h\, g_{jk}\right)$

Thus, we have

Theorem 2.8. In GR^h-FR-F_n, the tensor $\left(P^i_{jkh} + \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right)$ is $h - GFR$.

Transvecting (2.27) by y^j, using (1.1d), (1.7a), (1.14) and (1.6c), we get

(2.30) $H^i_{kh} = P^i_{kh} + \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)$.

Taking the $h-$ covariant derivative fourth for (2.30) with respect to x^ℓ, x^m, x^n and x^s, respectively, we get

(2.31) $H^i_{kh|\ell|m|n|s} = \left(P^i_{kh} + \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)\right)_{|\ell|m|n|s}$

Using condition (2.6) and (2.30) in (2.31), we get

$u_{\ell mns}\, H^i_{kh} + v_{\ell ms}\left(\delta^i_k\, y_h - \delta^i_h\, y_k\right) = \left(P^i_{kh} + \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)\right)_{|\ell|m|n|s}$

(2.32) $\left(P^i_{kh} + \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)\right)_{|\ell|m|n|s} = u_{\ell mns}\left(P^i_{kh} + \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)\right) + v_{\ell ms}\left(\delta^i_k\, y_h - \delta^i_h\, y_k\right)$

Transvecting (2.32) by the metric g_{ir}, using (1.1e), (1.4a) and (1.2b), we get

(2.33) $\left(P_{rkh} + \frac{1}{3}\left(g_{hr} R_k - g_{kr} R_h\right)\right)_{|\ell|m|n|s} = u_{\ell mns}\left(P_{rkh} + \frac{1}{3}\left(g_{hr} R_k - g_{kr} R_h\right)\right) + v_{\ell ms}\left(g_{kr}\, y_h - g_{hr}\, y_k\right)$

Thus , we have

Theorem 2.9. In GR^h-FR-F_n the tensors $\left(P^i_{kh} + \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)\right)$ and $\left(P_{rkh} + \frac{1}{3}\left(g_{hr} R_k - g_{kr} R_h\right)\right)$ are $h - GFR$ For $n = 4$.

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حول تعميمات الموتر R^h رباعي المعاودة في فضاء فنسلر

أماني محمد عبدالله حنبلة
قسم الرياضيات - كلية المجتمع - عدن

وفاء هادي علي هادي
قسم الرياضيات - كلية المجتمع - عدن

الملخص: في هذه الورقة، قدمنا تعريف للتعميم الرباعي لفضاء فنسلر F_n الذي يحقق الموتر التقوسي الثالث لكارتان R^i_{jkh} باستخدام مشتقة كارتان رباعية المعاودة وأطلقنا على هذا الفضاء بتعميم فضاء فنسلر R^h - رباعي المعاودة ورمزنا إليه بالرمز $GR^h - FR - \mathrm{F_n}$. ايضاً أوجدنا الشرط اللازم والكافي لبعض الموترات بمفهوم كارتان لكي تكون معممة رباعية المعاودة في هذا الفضاء.

الكلمات المفتاحية: فضاء فنسلر، تعميم R^h - رباعي المعاودة، موتر ريكي.