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Study of Generalized Fejèr Kernel of Fourier Series

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Abstract: In this article, we introduce p-Cesaro matrix, in order to generalized Fejer kernel based on p-integer. We study approximation continuous and periodic function by their Fourier series on interval [-L,L] . Finally, we obtain positive theorems about uniform convergent based on p-integer.

Keywords: Fourier Series, p-Cesaro Matrix , Fejer Kernel , p-integer.

1. Introduction

We are asking that of whether the Fourier series convergent to continuous function, we research by a field known as classical harmonic, analysis ([4],[9],).We know that convergence is not necessarily to given in the general case. After all, we study some summability approaches, just as p-Cesaro matrix and p- Fejer kernel, in order to get convergence of Fourier series(see [2],[13]). In the latest article, we use generalized Fejer kernel on p-integer, we study uniform convergence of Fourier series by Korovkin theorem on interval [-L,L].

First, we introduce some mathematical preliminaries that are important to studying uniform convergence of Fourier series(see[16]),

2. Preliminaries

Definition 2.1. Let $S_m f(x)$([4],[15]) of an integrable and 2L-Periodic function $f(x)$ such that

$$S_m f(x) = \frac{a_0}{2} + \sum_{k=1}^{m}\left(a_k cos\left(\frac{k\pi x}{L}\right) + b_k sin\left(\frac{k\pi x}{L}\right)\right), \qquad (2.1)$$

the coefficients are define by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt,$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(t) cos\frac{k\pi t}{L}\,dt, k \in N \cup \{0\} \text{ and}$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(t) sin\frac{k\pi t}{L}\,dt, \ k \in N.$$

The equation (2.1) is called Partial sums.

Definition 2.2. Let m≥ 0, ([3]) the Dirichlet's kernel defined by

$D_m(t) = \frac{1}{2} + \sum_{k=1}^{m} cos\left(\frac{k\pi t}{L}\right)$

$= \frac{1}{2}\sum_{k=-m}^{m} e^{\left(\frac{ik\pi t}{L}\right)}$, (2.2)

from (2.1) we get

$S_m f(x) = \frac{1}{L}\int_{-L}^{L} f(x-t)\, D_m(t)dt.$

The Partial sums $D_m(t)$ given by

$D_m(t) = \frac{sin\left(m+\frac{1}{2}\right)\frac{\pi t}{L}}{2sin\left(\frac{\pi t}{2L}\right)}$,

$sin\left(\frac{\pi t}{2L}\right) \neq 0$. (2.3)

Definition 2.3 Let m≥ 0,([4]) we define the Fejer kernel as

$F_m(t) = \frac{\sum_{j=0}^{m} D_j(t)}{2(m+1)}$

$= \frac{1}{2(m+1)}\left(\frac{sin\left[(m+1)\left(\frac{t\pi}{2L}\right)\right]}{sin\left(\frac{t\pi}{2L}\right)}\right)^2$

$sin\left(\frac{\pi t}{2L}\right) \neq 0$, (2.4)

$\sigma_m f(x) = \frac{1}{(m+1)}\sum_{j=0}^{m} S_j f(x)$,

then, the Cesaro sums of $\sigma_m f(x)$ is

$\sigma_m f(x) = \frac{1}{L}\int_{-L}^{L} F_n(t) f(x-t)dt$, (2.5)

where $F_m(t)$ is Fejer kernel.

Definition 2.4. ((Korovkin)[5], [11], [12]).

Let $Y = \{f \in C[-L,L]: f(L)f(-L)\}$.

Consider $f_1(t) = 1$, $f_2(t) = cos\left(\frac{\pi t}{L}\right)$, $f_3(t) = sin\left(\frac{\pi t}{L}\right)$ $for\ \ t \in [-L,L]$. Let $P_m: Y \to Y$ be a positive linear map for m=1,2,3,... if the sequence of functions $\left(P_m(f_j)\right)$ converges uniformly to the function f_j for j ∈ {1,2,3} on $[-L,L]$, then the sequence of functions $\left(P_m(f)\right)$ converges uniformly to the function f on $[-L,L]$ for all $f \in Y$.

Theorem 2.5. ((Abel′s summation), [14]).

Let two sums $\sum_{j=1}^{m} a_j$ and $\sum_{j=1}^{m} b_j$ denoted by $B_m = \sum_{i=1}^{m} a_i$ Hence .

$\sum_{j=n}^{m} a_j b_j = B_m b_m + B_n b_{n-1}$

$-\sum_{j=n}^{m-1} B_j\left(b_{j+1} - b_j\right)$, for $n \in N$.

Now, we define $C_{[-L,L]}^{2L}$ the space of all continuous and 2L periodic functions.

Hence, every $f \in C_{[-L,L]}^{2L}$, the sequence $\sigma_m f(x)$ be uniform convergence to $f(x)$ .we generalize Fejer kernel by p-integer..

3. P- Fejer kernel:

Now, we given some image and relations from the p-calculus (see [14] for details).

Let $p > 0$, the p-integer $[m]_p$ is denote by

$[m]_p = 1 + p + p^2 + \cdots + p^{n-1}$

where $[0]_p = 0$.

Hence , if m=1,2,.., the partial summation is

$[m]_p = \frac{1-p^m}{1-p}$, $\forall\, p \neq 1$.

Let $p > 0$, then the p-Cesaro matrix $C(p) = \left[c_{mj}(p)\right](j,m \in N \cup \{0\})$ denoted by ([8], [10], [15]).

$$c_{mj}(p) = \begin{cases} \frac{p^j}{[m+1]_p}, & j = 0,1,2,\dots m \\ 0, & otherwie\ . \end{cases} \quad (3.1)$$

Hence , we may moulding the matrix $C(p)$ following:

$$C(p) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \frac{1}{[2]_p} & \frac{p}{[2]_p} & 0 & \cdots & 0 & 0 & \cdots \\ \frac{1}{[3]_p} & \frac{p}{[3]_p} & \frac{p^2}{[3]_p} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{1}{[1+m]_p} & \frac{p}{[1+m]_p} & \frac{p^2}{[1+m]_p} & \cdots & \frac{p^n}{[1+m]_p} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

From $C(p)$, if p=1 then the p-Cesaro matrix in (3.1) write

$$c_{mj}(1) = \begin{cases} \frac{1}{m+1}, & m \in N \cup \{0\} \\ 0, & otherwie \end{cases} .$$

Now, we give some results.

● For each $p \geq 1$, the matrix $C(p)$ be regular (see **[8]).**

● For each $0 < p < 1$, the matrix $C(p)$ is not regular , because $\lim_{m\to\infty}[m+1]_p = \frac{1}{1-p}$.

● Replaces of a fixed p, give a sequence p=$\{p_m\}_{m\in N\cup\{0\}}$ the following is :

$$0 < p_m < 1 \text{ for all } m \in N \cup \{0\} . \qquad (3.2)$$

And

$$\lim_{m\to\infty} p_m = 1. \qquad (3.3)$$

Hence , from **(3.2)** and **(3.3),** then $C(p)$ is regular. We can moulding that

$\lim_{m\to\infty}[m+1]_{p_m} = \infty$. (see [6],[7],[8]).

Since the sequence of partial sums $S_m f(x)$ cannot convergence to $f(x)$, we can to give p-Cesaro mean :

$$\sigma_m f(p;x) = \frac{1}{[m+1]_p}\sum_{j=0}^{m} p^j S_j f(x) = \frac{S_0 f(x)+pS_1 f(x)+\cdots+p^m S_m f(x)}{[m+1]_p}.$$

This implies

$$\sigma_m f(p;x) = \frac{1}{L}\int_{-L}^{L} f(x-t)\left(\frac{1}{[m+1]_p}\sum_{j=0}^{m} p^j D_j(t)\right)dt.$$

Then, we can write

$$\sigma_m f(p;x) = \frac{1}{L}\int_{-L}^{L} f(x-t)F_m(p;t)dt, \qquad (3.4)$$

where $F_m(p;t)$ give by

$$F_m(p;t) = \frac{1}{2[m+1]_p}\sum_{j=0}^{m} p^j \frac{\sin\left(\left(m+\frac{1}{2}\right)\frac{\pi t}{L}\right)}{\sin\left(\frac{t\pi}{2L}\right)}. \qquad (3.5)$$

Called p-Fejer kernel.

From the p-Cesa'ro means in (3.4), it is a special case of Riesz means of the partial sums of Fourier series (see[1]), nevertheless, it is best to study convergence behaviour of the p-Fejer kernel in (3.5) with attention to the properties of the p-integers. More importantly for using p-integers, because generalized Fejer kernel and study converges uniformly [3].

Now, we give theorem of p-Fejer kernel.

Theorem 3.1. If $p > 0$ and $m \in N \cup \{0\}$. we have, the following:

i) $F_m(p;t) =$

$$\frac{(1+p)\sin\left(\frac{\pi t}{2L}\right)+p^{m+2}\sin\left(\left(m+\frac{1}{2}\right)\frac{\pi t}{L}\right)-p^{m+1}\sin\left(\left(m+\frac{3}{2}\right)\frac{\pi t}{L}\right)}{2[m+1]_p\sin\left(\frac{\pi t}{2L}\right)\left(1-2p\cos\left(\frac{\pi t}{L}\right)+p^2\right)}.$$

ii) $\frac{1}{L}\int_{-L}^{L} F_m(p;t)dt = \frac{m+1}{[m+1]_p}$.

iii) $if\ 0 < p \leq 1\ then,\ F_m(p;t) \geq 0$.

Proof.

i) From the equation (2.5), we can moulding that

$$F_m(p;t) = \frac{1}{2[m+1]_p}\sum_{j=0}^{m} p^j \frac{\sin\left(\left(j+\frac{1}{2}\right)\frac{\pi t}{L}\right)}{\sin\left(\frac{t\pi}{2L}\right)} = \frac{1}{2[m+1]_p\sin\left(\frac{t\pi}{2L}\right)} Im\left\{\sum_{j=0}^{m} p^j e^{i\left(j+\frac{1}{2}\right)\frac{\pi t}{L}}\right\},$$

$$= \frac{1}{2[m+1]_p\sin\left(\frac{t\pi}{2L}\right)} Im\left\{e^{\frac{i\pi t}{2L}}\sum_{j=0}^{m}\left(pe^{i\frac{\pi t}{L}}\right)^j\right\},$$

$$= \frac{1}{2[m+1]_p\sin\left(\frac{t\pi}{2L}\right)} Im\left\{e^{\frac{i\pi t}{2L}}.\frac{1-p^{m+1}e^{i(m+1)\frac{\pi t}{L}}}{1-pe^{\frac{i\pi t}{L}}}\right\},$$

$$= \frac{1}{2[m+1]_p\sin\left(\frac{t\pi}{2L}\right)} Im\left\{\frac{1-p^{m+1}e^{i(m+1)\frac{\pi t}{L}}}{e^{\frac{-i\pi t}{2L}}-pe^{\frac{i\pi t}{2L}}}\right\}.$$

The above equation can be written as

$F_m(p;t) = \frac{(1+p)sin\left(\frac{\pi t}{2L}\right)+p^{m+2}sin\left(\left(m+\frac{1}{2}\right)\frac{\pi t}{L}\right)-p^{m+1}sin\left(\left(m+\frac{3}{2}\right)\frac{\pi t}{L}\right)}{2[m+1]_p sin\left(\frac{\pi t}{2L}\right)\left(1-2pcos\left(\frac{\pi t}{L}\right)+p^2\right)}$

(ii) using Definition(2.5) in (3.5), we get:

$\frac{1}{L}\int_{-L}^{L} F_m(p;t)dt =$

$\frac{1}{L}\int_{-L}^{L} \frac{m+1}{[m+1]_p}\sum_{j=0}^{m} p^j \frac{sin\left(\left(j+\frac{1}{2}\right)\frac{\pi t}{L}\right)}{2(m+1)sin\left(\frac{\pi t}{2L}\right)}$

$= \frac{m+1}{[m+1]_p}\left\{p^m\left(\frac{1}{L}\int_{-L}^{L} F_m(t)dt\right) + (1-p)\sum_{j=0}^{m-1} p^j\left(\frac{1}{L}\int_{-L}^{L} F_j(t)dt\right)\right\}$

Since $\frac{1}{L}\int_{-L}^{L} F_m(t)dt$=1, we observe that

$\frac{1}{L}\int_{-L}^{L} F_m(p;t)dt = \frac{m+1}{[m+1]_p}\{p^n + (1-\ p)\sum_{j=0}^{m-1} p^j\}$

$= \frac{m+1}{[m+1]_p}\{p^m + (1-p)[m]_p\} = \frac{m+1}{[m+1]_p}$,

which complete the proof.

(iii) again from Definition(2.5), then we get that

$F_m(p;t) = \frac{(m+1)}{[m+1]_p}\sum_{j=0}^{m} p^j \frac{sin\left(\left(j+\frac{1}{2}\right)\frac{\pi t}{L}\right)}{2(m+1)sin\left(\frac{t\pi}{2L}\right)}$

$= \frac{(m+1)}{[m+1]_p}\{p^m F_m(t) + \sum_{j=0}^{m-1}(p^j - p^{j+1})F_j(t)\}$,

Hence , it is clear $F_m(p;t)$ be positive , since $0 < p \le 1$.

Remark .3.1. we take p=1 in **theorem 3.1** hence (i) implies

$F_m(1;t) = \frac{2sin\left(\frac{\pi t}{2L}\right)+sin\left(\left(m+\frac{1}{2}\right)\frac{\pi t}{L}\right)-sin\left(\left(m+\frac{3}{2}\right)\frac{\pi t}{L}\right)}{2(1+m)sin\left(\frac{\pi t}{2L}\right)\left(1-2cos\left(\frac{\pi t}{L}\right)+1\right)}$

$= \frac{sin^2\left(\left(\frac{m+1}{2}\right)\frac{\pi t}{L}\right)}{2(m+1)sin^2\left(\frac{\pi t}{2L}\right)} = F_m(t)$.

Meaning of that the same equality $F_m(1;t)$=$F_m(t)$, also (ii) and (iii) implies the classical results $\frac{1}{L}\int_{-L}^{L} F_m(t)dt$=1and $F_m(t) \ge 0$ respectively.

Now, we give uniform convergence of p-Fejer kernel.

Theorem 3.2. Let the sequence p=(p_m) satisfies in (3.3) and (3.4).Then $\forall\delta > 0$, $F_m(p_m;t) \underset{m\to\infty}{\longrightarrow} 0$, on $[-L,-\delta] \cup [\delta, L]$.

Proof. Given $\delta > 0$, we have that for all $\delta \le t \le L$, since $sin\left(\frac{\pi t}{2L}\right) \ge sin\left(\frac{\pi t}{2L}\right)$ and

$1-2pcos\left(\frac{\pi t}{L}\right)+p^2$

$= 1-2p+p^2-2pcos\left(\frac{\pi t}{L}\right)+p^2$,

$= (1-p)^2 + 2p\left(1-cos\frac{\pi t}{L}\right)$.

So, for all $\delta \le t \le L$:

$(1-p)^2 + 2p\left(1-cos\frac{\pi t}{L}\right) \ge (1-p)^2 +2p\left(1-cos\frac{\pi\delta}{L}\right)$, thus

$F_m(p_m;t) \le \frac{1+p_m}{2[m+1]_p sin\left(\frac{\delta\pi}{2L}\right)(1-p)^2+2p\left(1-cos\frac{\pi\delta}{L}\right)} + \frac{(p_m)^{m+2}}{2[m+1]_p sin\left(\frac{\delta\pi}{2L}\right)(1-p)^2+2p\left(1-cos\frac{\pi\delta}{L}\right)} + \frac{(p_m)^{m+3}}{2[m+1]_p sin\left(\frac{\delta\pi}{2L}\right)(1-p)^2+2p\left(1-cos\frac{\pi\delta}{L}\right)}$.

Notice that this tends to **o** where m→ ∞, independently of **t**, that is uniform convergent on $[-L,-\delta] \cup [\delta, L]$, which is the complete the proof.

Theorem 3.3. Suppose that the sequence p=(p_m) in **(3.3)** and **(3.4)**. Hence, in **(3.5),** be $\sigma_m f(q_m;x) \underset{m\to\infty}{\longrightarrow} f(x)$, $f(x) \in C_{[-L,L]}^{2L}$.

Proof. Since $\sigma_m f(p;x)$ is positive and linear, then using Definition 2.4 to prove that

$\sigma_m f_j(p_m;x) \rightrightarrows f_j(x)\ for\ j = 0,1,2,$ (3.6)

Using Definition.2.4 (Korovkin) be $f_0(x) = 1\ and f_1(x) = sin\frac{\pi x}{L}\ and\ f_2(x) = cos\frac{\pi x}{L}$, we chick $\sigma_m f_0(p_m;x)$ and $\sigma_m f_1(p_m;x) and\ \sigma_m f_2(p_m;x)$

$$\sigma_m f_0(p_m;x) = f_0(x) = 1. \qquad (3.7)$$

From the p-Fejer, we get

$$\sigma_m f_1(p_m;x) = \frac{S_0 f_1(x)+p_m S_1 f_1(x)+p_m^2 S_2 f_1(x)+\cdots+p_m^m S_m f_1(x)}{[m+1]_{p_m}},$$

$$= \frac{p_m sin\left(\frac{\pi x}{L}\right)+p_m^2 sin\left(\frac{\pi x}{L}\right)+\cdots+p_m^m sin\left(\frac{\pi x}{L}\right)}{[m+1]_{p_m}}$$

$$= \frac{p_m+p_m^2+\cdots+p_m^m}{[m+1]_{p_m}} sin\left(\frac{\pi x}{L}\right).$$ Hence

$$\sigma_m f_1(p_m;x) = \left(1-\frac{1}{[m+1]_{p_m}}\right) sin\left(\frac{\pi x}{L}\right). \qquad (3.8)$$

Likewise, we obtain

$$\sigma_m f_2(p_m;x) = \left(1-\frac{1}{[m+1]_{p_m}}\right) cos\left(\frac{\pi x}{L}\right). \qquad (3.9)$$

We take Limit where m→ ∞ and use (3.2) and (3.3) observe that

$$\lim_{m\to\infty} \sigma_m f_0(p_m;x) = 1.$$

$$\lim_{m\to\infty}\left(1-\frac{1}{[m+1]_{p_m}}\right) sin\left(\frac{\pi x}{L}\right) = sin\left(\frac{\pi x}{L}\right).$$

And

$$\lim_{m\to\infty}\left(1-\frac{1}{[m+1]_{p_m}}\right) cos\left(\frac{\pi x}{L}\right) = cos\left(\frac{\pi x}{L}\right).$$

We obtain

$$\sigma_m f_j(p_m;x) \rightrightarrows f_j(x),$$

which complete the proof.

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دراسة حول تعميم نواة فيجر من متسلسلة فورية

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الملخص: في هذا البحث، نقدم p-Cesaro Matrix من أجل تعميم نواة فيجر استناداً من p-integer . نحن ندرس التقريب المستمر من خلال سلسلة فورية على الفترة [-L,L]. اخيراً حصلنا على نظريات ايجابية حول التقارب المنتظم استناداً من p-integer .

الكلمات المفتاحية: متسلسلة فورية، مجموع المصفوفات لسيزارو ، نواة فيجر، p-integer