Exact Solutions for Two New (2+1) and (3+1)-Dimensional Clannish Random Walker's Parabolic Equation: $\left(\frac{1}{G''},\frac{G'}{G''}\right)$-Expansion Method

E. F. AL-Abdali [1*] **M. S. AL-Amry** [2]
[1,2] Department of Mathematics, Faculty of Education Aden, Aden University, Yemen
emanfadhel328@ gmail.com [1*] & salehm251251@gmail.com [2]

DOI: https://doi.org/10.47372/jef.(2026)20.1.217

Abstract: In this paper, we presented two new (2+1) and (3+1)-dimensional Clannish Random Walker's Parabolic (CRWP) equations of nonlinear partial differential equations (NLPDEs). We will use a new method, namely, the $\left(\frac{1}{G''},\frac{G'}{G''}\right)$-expansion method to conduct this analysis. Exact travelling wave solutions are obtained and expressed in terms of hyperbolic functions, trigonometric functions and rational functions.

Keywords: $\left(\frac{1}{G''},\frac{G'}{G''}\right)$-E xpansion Method, Clannish Random Walker's Parabolic Equation, Solitary wave Solutions.

1. Introduction: Nonlinear partial differential equations (NLPDEs) are used to simulate wide range of complicated processes arising in real life and all fields of science. Exact solutions of NLPDEs can be implemented to produce significant predictions for many applications related to nature and living. Modern theories of nonlinear science have been highly developed over the last half century. Particularly, integrable nonlinear systems have attracted great interest among a number of mathematicians and physicists. It is well known that the study of soliton generating nonlinear partial differential equations are of great interest in recent years not only in (1+1)-dimensional system, but also in (2+1)-dimensional system under certain conditions [1,2]. Generally speaking, analyzing coherent type soliton structures of (2+1)-dimensional system are more complicated than (1+1)-dimensional system. The study of higher-dimensional integrable systems is one of the main themes in integrability theory. Several models in the context of (2+1) -dimensional equations, which are integrable have been developed by Toda and Yu [3]. So, field of integrable equations is an active multidisciplinary area of research due to the fact that integrable equations describe the real features and reveal the mysterious nature of the nonlinearity in science and engineering applications [4]-[10].

In recent years, the study of integrable equations has become a hot research area in nonlinear mathematical physics and wave propagations. This originates from the fact that these equations give qualitative and quantitative features of several scientific aspects of many unrelated phenomena such as plasmas, fluids, lattice vibrations of a crystal at low temperature, propagation of waves in shallow water area, traffic flow, soliton propagation in nonlinear transmission lines, etc [11]-[17].

The (1+1)-dimensional CRWP equation reads [18]-[20]

$$u_t - u_x + u_{xx} + \alpha u u_x = 0. \tag{1}$$

In this work, we aim to presentation two new (2+1)-dimensional and (3+1)-dimensional CRWP equations, with constant coefficients. We first introduce a (2+1)-dimensional CRWP equation the form

$u_t - u_x + u_{xx} + \alpha\left(u\partial_y^{-1}u_x\right)_x = 0$, or equivalently

$$\begin{aligned} u_t - u_x + u_{xx} + \alpha\left(uv\right)_x &= 0, \\ v_y &= u_x. \end{aligned} \tag{2}$$

We introduce a (3+1)-dimensional CRWP equation with constant coefficients takes the form

$$u_t - u_x + \alpha u u_y + \beta u_x \partial_y^{-1} u_x + \gamma u_z \partial_y^{-1} u_z + \varepsilon\left(\alpha u_{yy} + \beta u_{xx} + \gamma u_{zz}\right) = 0,$$

or equivalently

$$u_t - u_x + \alpha u u_y + \beta v u_x + \gamma w u_z + \varepsilon\left(\alpha u_{yy} + \beta u_{xx} + \gamma u_{zz}\right) = 0, \quad v_y = u_x, w_y = u_z. \tag{3}$$

Where α, β, and γ are constants.

2. The New $\left(\frac{1}{G''}, \frac{G'}{G''}\right)$-Expansion Method: In this section, we describe our new method, namely, the $\left(\frac{1}{G''}, \frac{G'}{G''}\right)$-expansion method for finding travelling wave solutions of nonlinear evolution equations. As preparations, consider the third order ordinary linear differential equation:

$$G''' + \lambda G' + \mu = 0, \tag{4}$$

and defining for the simplicity

$$\varphi(\xi) = \frac{1}{G''}, \psi(\xi) = \frac{G'}{G''}, \tag{5}$$

and

$$\varphi' = \mu\varphi^2 + \lambda\varphi\psi, \quad \psi' = \lambda\psi^2 + \mu\varphi\psi + 1. \tag{6}$$

Using the general solutions of the Eq. (4), we have the following solutions:

Case 1. when $\lambda < 0$, the general solutions of the Eq. (4) is

$$G(\xi) = C_1 \cosh\left(\sqrt{-\lambda}(\xi + h)\right) + C_2 \sinh\left(\sqrt{-\lambda}(\xi + h)\right) - \frac{\mu}{\lambda}\xi, \tag{7}$$

and we have

$$\varphi^2 = \frac{-\lambda}{\lambda^3 \nu + \mu^2}\left(\lambda\psi^2 + 2\mu\varphi\psi + 1\right), \tag{8}$$

where C_1 and C_2 are two arbitrary constants and $\nu = C_2^2 - C_1^2$.

Case 2. when $\lambda > 0$, the general solutions of the Eq. (4) is

$$G(\xi) = C_1 \cos\left(\sqrt{\lambda}(\xi + h)\right) + C_2 \sin\left(\sqrt{\lambda}(\xi + h)\right) - \frac{\mu}{\lambda}\xi, \tag{9}$$

and we have

$$\varphi^2 = \frac{\lambda}{\lambda^3 \nu - \mu^2}\left(\lambda\psi^2 + 2\mu\varphi\psi + 1\right), \tag{10}$$

where C_1 and C_2 are two arbitrary constants and $\nu = C_2^2 + C_1^2$.

Case 3. when $\lambda = 0$, the general solutions of the Eq. (4) is

$$G(\xi) = -\frac{\mu}{6}\xi^3 + \frac{1}{2}C_1\xi^2 + C_2\xi + C_1, \tag{11}$$

and we have

$$\varphi^2 = \frac{1}{C_1^2 + 2\mu C_2}\left(2\mu\varphi\psi + 1\right), \tag{12}$$

where C_1, C_2 and C_3 are arbitrary constants. Now consider the general nonlinear partial differential equations (NLPDEs), say, in two variables,

$$p\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\right) = 0, \tag{13}$$

where $u = u(x, t)$ is an unknown function, p is a polynomial in $u(x, t)$ and the subscripts stand for the partial derivatives?

We suppose that the combination of real variables x and t by a complex variable ξ

$$u(x,t)=u(\xi),\quad \xi=ax-ct, \tag{14}$$

where a is the wave number and c is the speed of the traveling wave. Now using Eq. (14), Eq. (13) is converted into an ordinary differential equation for $u=u(\xi)$:

$$F\left(u,-cu',au',c^2u'',-cau'',a^2u'',\ldots\right)=0,'\equiv\frac{d}{d\xi}. \tag{15}$$

Suppose that the traveling wave solution of Eq. (15) can be expressed by a polynomial in φ and ψ as:

$$u(\xi)=\sum_{i=0}^{N}a_i\psi^i+\sum_{i=1}^{N}b_i\psi^{i-1}\varphi, \tag{16}$$

where $G=G(\xi)$ satisfies the third order LODE (4), $a_i\ (i=0,1,2,\ldots,N)$, $b_i\ (i=0,1,2,\ldots,N),a,c,\lambda$ and μ are arbitrary constants and the positive integer N can be determined by using homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (15).
Substituting Eq. (16) into Eq. (15), using Eq. (6) and Eq. (8) (or using Eq. (6), Eq. (10) and Eq. (6), Eq. (12)) the left-hand side of Eq. (15) can be converted into a polynomial in ψ and φ, in which the degree of φ is not larger than 1. Compare the like powers of ψ^N and $\psi^N\varphi$ equal to zero, yields a set of algebraic equations for $a_i\ (i=0,1,2,\ldots,N),b_i\ (i=0,1,2,\ldots,N),a,c,\lambda,\mu,C_1$ and C_2. Solve the algebraic equations with the aid of Maple. Substituting the values of $a_i\ (i=0,1,2,\ldots,N)$, $b_i\ (i=0,1,2,\ldots,N),a,c,\lambda,\mu,C_1$ and C_2 obtained into Eq. (16), one can obtain the travelling wave solutions expressed by the hyperbolic functions of Eq. (15) (or expressed by trigonometric functions and rational functions).

3. (2+1)-dimensional CRWP equations: In this section, we aim first to apply our new method for solve Eq. (2).

Substituting $u(x,y,t)=u(\xi),\xi=ax+by-ct$ in Eq. (2) and integrating once yields

$$-(c+a)u(\xi)+\frac{a^2\alpha}{b}\left(u(\xi)\right)^2+a^2u'(\xi)+k=0. \tag{17}$$

Balancing Eq. (17) give:
$2N=N+1\Rightarrow N=1.$
By the use of Eq. (16), we present the solution of Eq. (17) as:

$$u(\xi)=a_0+a_1\psi(\xi)+b_1\varphi(\xi). \tag{18}$$

Case 1. When $\lambda<0$ (hyperbolic function solutions)
Substituting Eq. (18) into Eq. (17), using Eq. (6) and Eq. (8) the left-hand side of Eq. (17) becomes a polynomial in ψ and φ. Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0,a_1,b_1,a,b,c,\alpha,\mu$ and λ. After solving the algebraic equations by Maple we get,

set 1.

$$a_0=a_0,a_1=\frac{-b\lambda}{\alpha},b_1=\frac{-b\mu}{\alpha},c=\frac{a\left(2a\alpha a_0-b\right)}{b},$$

set 2.

$$a_0=a_0,a_1=\frac{-b\lambda}{2\alpha},b_1=\frac{b\left(-\mu\pm\lambda\sqrt{-\left(c_2^2-c_1^2\right)\lambda}\right)}{2\alpha},c=\frac{a\left(2a\alpha a_0-b\right)}{b}.$$

Putting the set (1) into Eq. (18), we get

$$u_1=a_0+\frac{b\sqrt{-\lambda}\left(C_1\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)}{\alpha\left(C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)},$$

$$v_1=\frac{aa_0}{b}+\frac{a\sqrt{-\lambda}\left(C_1\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)}{\alpha\left(C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)}.$$

If $C_1\neq 0$ and $C_2=0$,

$$u_2=a_0+\frac{b\sqrt{-\lambda}}{\alpha}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right),$$

$$v_2=\frac{aa_0}{b}+\frac{a\sqrt{-\lambda}}{\alpha}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right).$$

If $C_2\neq 0$ and $C_1=0$,

$$u_3=a_0+\frac{b\sqrt{-\lambda}}{\alpha}\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right),$$

$$v_3=\frac{aa_0}{b}+\frac{a\sqrt{-\lambda}}{\alpha}\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right).$$

Putting the set (2) into Eq. (18), we get

$$u_{4,5}=a_0+\frac{b\left(C_1\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\sqrt{-\left(c_2^2-c_1^2\right)\lambda}\right)}{2\alpha\left(C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)},$$

$$v_{4,5}=\frac{aa_0}{b}+\frac{a\left(C_1\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\sqrt{-\left(c_2^2-c_1^2\right)\lambda}\right)}{2\alpha\left(C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)}.$$

If $C_1\neq 0$ and $C_2=0$,

$$u_{6,7}=a_0+\frac{b\sqrt{-\lambda}}{2\alpha}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\frac{b\sqrt{\lambda}}{2\alpha}\operatorname{sech}\left(\sqrt{-\lambda}\left(\xi+h\right)\right),$$

$$v_{6,7}=\frac{aa_0}{b}+\frac{a\sqrt{-\lambda}}{2\alpha}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\frac{a\sqrt{\lambda}}{2\alpha}\operatorname{sech}\left(\sqrt{-\lambda}\left(\xi+h\right)\right).$$

If $C_2\neq 0$ and $C_1=0$,

$$u_{8,9}=a_0+\frac{b\sqrt{-\lambda}}{2\alpha}\left(\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\operatorname{csch}\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right),$$

$$v_{8,9}=\frac{aa_0}{b}+\frac{a\sqrt{-\lambda}}{2\alpha}\left(\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\operatorname{csch}\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right).$$

Case 2. When $\lambda>0$ (trigonometric function solutions)

Substituting Eq. (18) into Eq. (17), using Eq. (6) and Eq. (10) the left-hand side of Eq. (17) becomes a polynomial in ψ and φ. Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0,a_1,b_1,a,b,c,\alpha,\mu$ and λ. After solving the algebraic equations by Maple we get,

set 1.

$$a_0=a_0,a_1=\frac{-b\lambda}{\alpha},b_1=\frac{-b\mu}{\alpha},c=\frac{a\left(2a\alpha a_0-b\right)}{b},$$

set 2.

$$a_0=a_0,a_1=\frac{-b\lambda}{2\alpha},b_1=\frac{b\left(-\mu\pm\lambda\sqrt{\left(c_2^2+c_1^2\right)\lambda}\right)}{2\alpha},c=\frac{a\left(2a\alpha a_0-b\right)}{b}.$$

Putting the set (1) into Eq. (18), we get

$$u_1 = a_0 + \frac{b\sqrt{\lambda}\left(-C_1 \sin\left(\sqrt{\lambda}(\xi+h)\right)+C_2 \cos\left(\sqrt{\lambda}(\xi+h)\right)\right)}{\alpha\left(C_1 \cos\left(\sqrt{\lambda}(\xi+h)\right)+C_2 \sin\left(\sqrt{\lambda}(\xi+h)\right)\right)},$$

$$v_1 = \frac{aa_0}{b} + \frac{a\sqrt{\lambda}\left(-C_1 \sin\left(\sqrt{\lambda}(\xi+h)\right)+C_2 \cos\left(\sqrt{\lambda}(\xi+h)\right)\right)}{\alpha\left(C_1 \cos\left(\sqrt{\lambda}(\xi+h)\right)+C_2 \sin\left(\sqrt{\lambda}(\xi+h)\right)\right)}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_2 = a_0 - \frac{b\sqrt{\lambda}}{\alpha}\tan\left(\sqrt{\lambda}(\xi+h)\right),$$

$$v_2 = \frac{aa_0}{b} - \frac{a\sqrt{\lambda}}{\alpha}\tan\left(\sqrt{\lambda}(\xi+h)\right).$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_3 = a_0 + \frac{b\sqrt{\lambda}}{\alpha}\cot\left(\sqrt{\lambda}(\xi+h)\right),$$

$$v_3 = \frac{aa_0}{b} + \frac{a\sqrt{\lambda}}{\alpha}\cot\left(\sqrt{\lambda}(\xi+h)\right).$$

Putting the set (2) into Eq. (18), we get

$$u_{4,5} = a_0 + \frac{b\left(-C_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}(\xi+h)\right)+C_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}(\xi+h)\right)\mp\sqrt{\left(c_2^2+c_1^2\right)\lambda}\right)}{2\alpha\left(C_1 \cos\left(\sqrt{\lambda}(\xi+h)\right)+C_2 \sin\left(\sqrt{\lambda}(\xi+h)\right)\right)},$$

$$v_{4,5} = \frac{aa_0}{b} + \frac{a\left(-C_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}(\xi+h)\right)+C_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}(\xi+h)\right)\mp\sqrt{\left(c_2^2+c_1^2\right)\lambda}\right)}{2\alpha\left(C_1 \cos\left(\sqrt{\lambda}(\xi+h)\right)+C_2 \sin\left(\sqrt{\lambda}(\xi+h)\right)\right)}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_{6,7} = a_0 - \frac{b\sqrt{\lambda}}{2\alpha}\left(\tan\left(\sqrt{\lambda}(\xi+h)\right)\mp\sec\left(\sqrt{\lambda}(\xi+h)\right)\right),$$

$$v_{6,7} = \frac{aa_0}{b} - \frac{a\sqrt{\lambda}}{2\alpha}\left(\tan\left(\sqrt{\lambda}(\xi+h)\right)\mp\sec\left(\sqrt{\lambda}(\xi+h)\right)\right).$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_{8,9} = a_0 + \frac{b\sqrt{\lambda}}{2\alpha}\left(\cot\left(\sqrt{\lambda}(\xi+h)\right)\mp\csc\left(\sqrt{\lambda}(\xi+h)\right)\right),$$

$$v_{8,9} = \frac{aa_0}{b} + \frac{a\sqrt{\lambda}}{2\alpha}\left(\cot\left(\sqrt{\lambda}(\xi+h)\right)\mp\csc\left(\sqrt{\lambda}(\xi+h)\right)\right).$$

Where $\xi = ax + by - \left(\frac{a(2a\alpha a_0 - b)}{b}\right)t$.

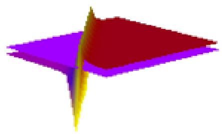
Case 3. When $\lambda = 0$ (rational function solutions)

Substituting Eq. (18) into Eq. (17), using Eq. (6) and Eq. (12) the left-hand side of Eq. (17) becomes a polynomial in ψ and φ. Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, a, b, c, \alpha, \mu$ and λ. After solving the algebraic equations by Maple we get,

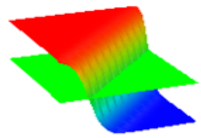
$$a_0 = 0, a_1 = 0, b_1 = \frac{-b\mu}{\alpha}, c = -a.$$

Putting the values of the constants into Eq. (18), we get

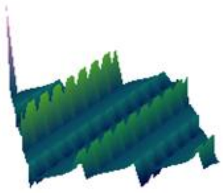
$$u_1 = \frac{-b\mu}{\alpha\left(-\mu\left(ax+by+at\right)+C_1\right)},$$
$$v_1 = \frac{-a\mu}{\alpha\left(-\mu\left(ax+by+at\right)+C_1\right)}.$$



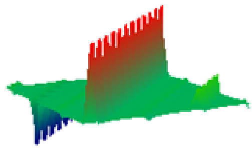
$u_1(x,y,t)$ if $\lambda<0$



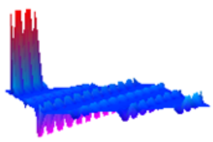
$u_2(x,y,t)$ if $\lambda<0$



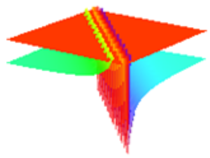
$u_1(x,y,t)$ if $\lambda>0$



$u_2(x,y,t)$ if $\lambda>0$



$u_8(x,y,t)$ if $\lambda>0$



$u_1(x,y,t)$ if $\lambda=0$

Fig 1. $3D$ -plot for $u_1(x,y,t)$, $u_2(x,y,t)$ when $\lambda<0$, $u_1(x,y,t)$, $u_2(x,y,t)$ and $u_8(x,y,t)$, when $\lambda>0$ and $\alpha=a=b=a_0=1, c_1=2, c_2=3, y=1.5, h=0.5$. $u_1(x,y,t)$ when $\lambda=0$, $\alpha=a=b=\mu=1, c_1=2$ and $y=2$.

4. (3+1)-dimensional CRWP equations: In this section, we apply our new method for solve Eq. (3). Substituting $u(x,y,z,t)=u(\xi), \xi=x+ay+bz-ct$ in Eq. (3) and integrating once yields

$$-(c+1)u(\xi)+\frac{1}{2}\left(a\alpha+\frac{\beta}{a}+\frac{\gamma b^2}{a}\right)\left(u(\xi)\right)^2+\varepsilon\left(\alpha a^2+\beta+\gamma b^2\right)u'(\xi)+k=0. \qquad (19)$$

Balancing Eq. (19) give:
$2N=N+1 \Rightarrow N=1.$
By the use of Eq. (16), we present the solution of Eq. (19) as:

$$u(\xi)=a_0+a_1\psi(\xi)+b_1\varphi(\xi). \qquad (20)$$

Case 1. When $\lambda<0$ (hyperbolic function solutions)
Substituting Eq. (20) into Eq. (19), using Eq. (6) and Eq. (8) the left-hand side of Eq. (19) becomes a polynomial in ψ and φ. Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, a, b, c, \alpha, \beta, \gamma, \mu$ and λ. After solving the algebraic equations by Maple we get,

set 1.

$$a_0=a_0,a_1=-2a\lambda\varepsilon,b_1=-2a\mu\varepsilon,c=\frac{a^2\alpha a_0+b^2\gamma a_0+\beta a_0-a}{a},$$

set 2.

$$a_0=a_0,a_1=-a\lambda\varepsilon,b_1=a\varepsilon\left(-\mu\pm\lambda\sqrt{-\left(c_2^2-c_1^2\right)\lambda}\right),$$

$$c=\frac{a^2\alpha a_0+b^2\gamma a_0+\beta a_0-a}{a}.$$

Putting the set (1) into Eq. (20), we get

$$u_1=a_0+\frac{2a\varepsilon\sqrt{-\lambda}\left(C_1\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)}{C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)},$$

$$v_1=\frac{a_0}{a}+\frac{2\varepsilon\sqrt{-\lambda}\left(C_1\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)}{C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)},$$

$$w_1=\frac{ba_0}{a}+\frac{2b\varepsilon\sqrt{-\lambda}\left(C_1\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right)}{C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)}.$$

If $C_1\neq 0$ and $C_2=0$,

$$u_2=a_0+2a\varepsilon\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right),$$

$$v_2=\frac{a_0}{a}+2\varepsilon\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right),$$

$$w_2=\frac{ba_0}{a}+2b\varepsilon\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right).$$

If $C_2\neq 0$ and $C_1=0$,

$$u_3=a_0+2a\varepsilon\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right),$$

$$v_3=\frac{a_0}{a}+2\varepsilon\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right),$$

$$w_3=\frac{ba_0}{a}+2b\varepsilon\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right).$$

Putting the set (2) into Eq. (20), we get

$$u_{4,5}=a_0+\frac{a\varepsilon\left(\sqrt{-\lambda}C_1\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+\sqrt{-\lambda}C_2\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\sqrt{-\left(c_2^2-c_1^2\right)\lambda}\right)}{C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)},$$

$$v_{4,5}=\frac{a_0}{a}+\frac{\varepsilon\left(\sqrt{-\lambda}C_1\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+\sqrt{-\lambda}C_2\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\sqrt{-\left(c_2^2-c_1^2\right)\lambda}\right)}{C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)},$$

$$w_{4,5}=\frac{ba_0}{a}+\frac{b\varepsilon\left(\sqrt{-\lambda}C_1\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+\sqrt{-\lambda}C_2\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\sqrt{-\left(c_2^2-c_1^2\right)\lambda}\right)}{C_1\cosh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)}.$$

If $C_1\neq 0$ and $C_2=0$,

$$u_{6,7}=a_0+a\varepsilon\left(\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\sqrt{\lambda}\operatorname{sech}\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right),$$

$$v_{6,7}=\frac{a_0}{a}+\varepsilon\left(\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\sqrt{\lambda}\operatorname{sech}\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right),$$

$$w_{6,7}=\frac{ba_0}{a}+b\varepsilon\left(\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\sqrt{\lambda}\operatorname{sech}\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right).$$

If $C_2\neq 0$ and $C_1=0$,

$$u_{8,9}=a_0+a\varepsilon\sqrt{-\lambda}\left(\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\operatorname{csch}\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right),$$

$$v_{8,9}=\frac{a_0}{a}+\varepsilon\sqrt{-\lambda}\left(\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\operatorname{csch}\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right),$$

$$w_{8,9}=\frac{ba_0}{a}+b\varepsilon\sqrt{-\lambda}\left(\coth\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\mp\operatorname{csch}\left(\sqrt{-\lambda}\left(\xi+h\right)\right)\right).$$

Case 2. When $\lambda>0$ (trigonometric function solutions)

Substituting Eq. (20) into Eq. (19), using Eq. (6) and Eq. (10) the left-hand side of Eq. (19) becomes a polynomial in ψ and φ. Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0,a_1,b_1,a,b,c,\alpha,\beta,\gamma,\mu$ and λ. After solving the algebraic equations by Maple we get,

set 1.

$$a_0=a_0,a_1=-2a\lambda\varepsilon,b_1=-2a\mu\varepsilon,c=\frac{a^2\alpha a_0+b^2\gamma a_0+\beta a_0-a}{a},$$

set 2.

$$a_0=a_0,a_1=-a\lambda\varepsilon,b_1=a\varepsilon\left(-\mu\pm\lambda\sqrt{\left(c_2^2+c_1^2\right)\lambda}\right),$$

$$c=\frac{a^2\alpha a_0+b^2\gamma a_0+\beta a_0-a}{a}.$$

Putting the set (1) into Eq. (20), we get

$$u_1=a_0+\frac{2a\varepsilon\sqrt{\lambda}\left(-C_1\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right)}{C_1\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)},$$

$$v_1=\frac{a_0}{a}+\frac{2\varepsilon\sqrt{\lambda}\left(-C_1\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right)}{C_1\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)},$$

$$w_1=\frac{ba_0}{a}+\frac{2b\varepsilon\sqrt{\lambda}\left(-C_1\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right)}{C_1\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)}.$$

If $C_1\neq 0$ and $C_2=0$,

$$u_2=a_0-2a\varepsilon\sqrt{\lambda}\tan\left(\sqrt{\lambda}\left(\xi+h\right)\right),$$

$$v_2=\frac{a_0}{a}-2\varepsilon\sqrt{\lambda}\tan\left(\sqrt{\lambda}\left(\xi+h\right)\right),$$

$$w_2=\frac{ba_0}{a}-2b\varepsilon\sqrt{\lambda}\tan\left(\sqrt{\lambda}\left(\xi+h\right)\right).$$

If $C_2\neq 0$ and $C_1=0$,

$$u_3=a_0+2a\varepsilon\sqrt{\lambda}\cot\left(\sqrt{\lambda}\left(\xi+h\right)\right),$$

$$v_3=\frac{a_0}{a}+2\varepsilon\sqrt{\lambda}\cot\left(\sqrt{\lambda}\left(\xi+h\right)\right),$$

$$w_3=\frac{ba_0}{a}+2b\varepsilon\sqrt{\lambda}\cot\left(\sqrt{\lambda}\left(\xi+h\right)\right).$$

Putting the set (2) into Eq. (20), we get

$$u_{4,5}=a_0+\frac{a\varepsilon\left(-\sqrt{\lambda}C_1\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)+\sqrt{\lambda}C_2\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\sqrt{\left(c_2^2+c_1^2\right)\lambda}\right)}{C_1\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)},$$

$$v_{4,5}=\frac{a_0}{a}+\frac{\varepsilon\left(-\sqrt{\lambda}C_1\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)+\sqrt{\lambda}C_2\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\sqrt{\left(c_2^2+c_1^2\right)\lambda}\right)}{C_1\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)},$$

$$w_{4,5}=\frac{ba_0}{a}+\frac{b\varepsilon\left(-\sqrt{\lambda}C_1\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)+\sqrt{\lambda}C_2\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\sqrt{\left(c_2^2+c_1^2\right)\lambda}\right)}{C_1\cos\left(\sqrt{\lambda}\left(\xi+h\right)\right)+C_2\sin\left(\sqrt{\lambda}\left(\xi+h\right)\right)}.$$

If $C_1\neq 0$ and $C_2=0$,

$$u_{6,7}=a_0-a\varepsilon\sqrt{\lambda}\left(\tan\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\sec\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right),$$

$$v_{6,7}=\frac{a_0}{a}-\varepsilon\sqrt{\lambda}\left(\tan\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\sec\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right),$$

$$w_{6,7}=\frac{ba_0}{a}-b\varepsilon\sqrt{\lambda}\left(\tan\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\sec\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right).$$

If $C_2\neq 0$ and $C_1=0$,

$$u_{8,9}=a_0+a\varepsilon\sqrt{\lambda}\left(\cot\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\csc\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right),$$

$$v_{8,9}=\frac{a_0}{a}+\varepsilon\sqrt{\lambda}\left(\cot\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\csc\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right),$$

$$w_{8,9}=\frac{ba_0}{a}+b\varepsilon\sqrt{\lambda}\left(\cot\left(\sqrt{\lambda}\left(\xi+h\right)\right)\mp\csc\left(\sqrt{\lambda}\left(\xi+h\right)\right)\right).$$

Where $\xi=x+ay+bz-\left(\frac{a^2\alpha a_0+b^2\gamma a_0+\beta a_0-a}{a}\right)t$.

Case 3. When $\lambda=0$ (rational function solutions)

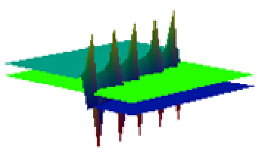
Substituting Eq. (20) into Eq. (19), using Eq. (6) and Eq. (12) the left-hand side of Eq. (19) becomes a polynomial in ψ and φ. Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0,a_1,b_1,a,b,c,\alpha,\beta,\gamma,\mu$ and λ. After solving the algebraic equations by Maple we get,

$a_0=0,a_1=0,b_1=-2a\varepsilon\mu,c=-1.$

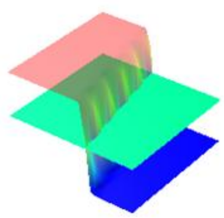
Putting the values of the constants into Eq. (20), we get

$$u_1=\frac{-2a\varepsilon\mu}{-\mu\left(x+ay+bz+t\right)+C_1},\qquad v_1=\frac{-2\varepsilon\mu}{-\mu\left(x+ay+bz+t\right)+C_1},$$

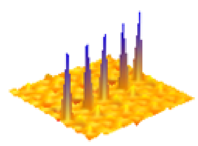
$$w_1=\frac{-2b\varepsilon\mu}{-\mu\left(x+ay+bz+t\right)+C_1}.$$



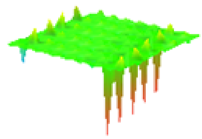
$u_1(x,y,z,t)$ if $\lambda<0$



$u_2(x,y,z,t)$ if $\lambda<0$



$u_1(x,y,z,t)$ if $\lambda > 0$



$u_2(x,y,z,t)$ if $\lambda > 0$

Fig 2. $3D$ -plot for $u_1(x,y,z,t)$, $u_2(x,y,z,t)$, when $\lambda < 0$, $u_1(x,y,z,t)$, $u_2(x,y,z,t)$, when $\lambda > 0$ and $\varepsilon = a = b = a_0 = y = 1, \alpha = \beta = \gamma = 2, c_1 = 2, c_2 = 3,$
$z = 0.5, h = 0$.

5. Conclusion: In this paper, we established two equations, namely, the (2+1)-dimensional Clannish Random Walker's Parabolic equation and the (3+1)-dimensional Clannish Random Walker's Parabolic equation, which we believe that these two equations are introduced to the first time. The $\left(\frac{1}{G''}, \frac{G'}{G''}\right)$ - expansion method has been successfully implemented to find the new two traveling waves solutions. The results show that this method is a powerful Mathematical tool for obtaining exact solutions for our equations. It is also a promising method to solve other nonlinear partial differential equations.

References

1. Konopelchenko, Boris Georgievich. Introduction to multidimensional integrable equations: the inverse spectral transform in (2+1)-dimensions. Springer Science & Business Media, (2013).
2. Boiti, Marco, et al. "Scattering of localized solitons in the plane." Physics Letters A 132.8-9 (1988): 432-439.
3. Toda, Kouichi, and Song-Ju Yu. "The investigation into the Schwarz-Korteweg-de Vries equation and the Schwarz derivative in (2+1)-dimensions." Journal of Mathematical Physics 41.7 (2000): 4747-4751.
4. Boiti, Marco, et al. "On the spectral transform of a Korteweg-de Vries equation in two spatial dimensions." Inverse problems 2.3 (1986): 271.
5. Wazwaz, Abdul-Majid. "Single and multiple-soliton solutions for the (2+1)- dimensional KdV equation." Applied mathematics and computation 204.1 (2008): 20-26.
6. Wazwaz, Abdul-Majid. "Painlevé analysis for new (3+ 1)-dimensional Boiti-Leon-Manna-Pempinelli equations with constant and time-dependent coefficients." International Journal of Numerical Methods for Heat & Fluid Flow 30.9 (2020): 4259-4266.
7. Wazwaz, Abdul-Majid. "Two new Painlevé-integrable (2+1) and (3+1)- dimensional KdV equations with constant and time-dependent coefficients." Nuclear Physics B 954 (2020): 115009.
8. Xu, Gui-Qiong, and Abdul-Majid Wazwaz. "Integrability aspects and localized wave solutions for a new (4+1)-dimensional Boiti-Leon-Manna-Pempinelli equation." Nonlinear Dynamics 98.2 (2019): 1379-1390.
9. He, Yinghui. "Exact Solutions for (4+1)-Dimensional Nonlinear Fokas Equation Using Extended F-Expansion Method and Its Variant." Mathematical Problems in Engineering 2014.1 (2014): 972519.
10. Khalique, Chaudry Masood. "Solutions and conservation laws of Benjamin--Bona--Mahony-Peregrine equation with power-law and dual power-law nonlinearities." Pramana 80 (2013): 413-427.
11. Al-Amry, M. S., and E. F. Abdullah. "Exact solutions for a new model of nonlinear partial differential equations Using $\left(G'/G^2\right)$-Expansion Method." University of Aden Journal of Natural and Applied Sciences 23.1 (2019): 189-199.
12. Al-Amry, M. S., and E. F. Al-Abdali. "New Exact Solutions for Generalized of Combined with Negative Calogero-Bogoyavlenskii Schiff and Generalized Yu-Toda-Sassa-Fukuyama Equations." University of Aden Journal of Natural and Applied Sciences 28.1 (2024): 25-30.
13. Leblond, Hervé, and Dumitru Mihalache. "Models of few optical cycle solitons beyond the slowly varying envelope approximation." Physics Reports 523.2 (2013): 61-126.
14. Wu, Guojiang, and Yong Guo. "New complex wave solutions and diverse wave structures of the (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equation." Fractal and Fractional 7.2 (2023): 170.

15. Adem, Khadijo Rashid, and Chaudry Masood Khalique. "Exact solutions and conservation laws of Zakharov-Kuznetsov modified equal width equation with power law nonlinearity." Nonlinear Analysis: Real World Applications 13.4 (2012): 1692-1702.
16. Wazwaz, Abdul-Majid. "A variety of (3+1)-dimensional mKdV equations derived by using the mKdV recursion operator." Computers & Fluids 93 (2014): 41-45.
17. Osman, M. S., and J. A. T. Machado. "The dynamical behavior of mixed-type soliton solutions described by (2+1)-dimensional Bogoyavlensky-Konopelchenko equation with variable coefficients." Journal of Electromagnetic Waves and Applications 32.11 (2018): 1457-1464.
18. Unal, Sevil Culha. "Approximate Solutions of the Fractional Clannish Random Walker's Parabolic Equation with the Residual Power Series Method." Journal of New Theory 45 (2023): 18-29.
19. Zafar, Asim, et al. "New fractional solutions for the Clannish Random Walker's Parabolic equation and the Ablowitz-Kaup-Newell-Segur equation." Optical and Quantum Electronics 56.4 (2024): 581.
20. Alam, Md Nur, et al. "Regarding on the Results for the Fractional Clannish Random Walker's Parabolic Equation and the Nonlinear Fractional Cahn-Allen Equation." Advances in Mathematical Physics 2022.1 (2022): 5635514.

حلول دقيقة لمعادلتين جديدتين في البُعدين الثاني والثالث من معادلة كلانش راندم ووكرز المكافئة: طريقة $\left(\frac{1}{G''}, \frac{G'}{G''}\right)$ الموسعة

إيمان فضل عبد الله أحمد العبدلي [1]
محمد سالم أحمد العمري [2]
قسم الرياضيات- كلية التربية عدن- جامعة عد ن

الملخص: في هذا البحث قدمنا صيغتين جديدتين من معادلة كلانش راندم ووكرز المكافئة (CRWP) ثنائية وثلاثية الأبعاد. ثم قمنا بتطبيق طريقة $\left(\frac{1}{G''}, \frac{G'}{G''}\right)$ الموسعة الجديدة لحل المعادلتين الجديدتين المقدمة. حيث تم الحصول على العديد من الحلول الدقيقة للموجات المتنقلة التي تم التعبير عنها بواسطة الدوال الزائدية، الدوال المثلثية والدوال الكسرية لهذه المعادلتين وذلك بمساعدة برنامج Maple. وأظهرت النتائج أنّ هذه الطريقة هي أداة رياضية قوية للحصول على حلول دقيقة لمعادلاتنا. وهي أيضا طريقة واعدة لحل المعادلات الجزئية غير الخطية الأخرى.

الكلمات المفتاحية: معادلة كلانش راندم ووكرز المكافئة - حلول الموجة المنفردة - طريقة $\left(\frac{1}{G''}, \frac{G'}{G''}\right)$ الموسعة