A New Transformation Rule Between Finsler Metrics and Its Geometric Implications on Tangent Structures and Conformal Invariants

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DOI: https://doi.org/10.47372/jef.(2026)20.1.194

Abstract: This paper introduces a new transformation rule between Finsler metrics and investigates its geometric implications within the framework of conformal Finsler geometry. Finsler geometry extends Riemannian geometry by allowing metric dependence on both positional and directional variables, providing a suitable setting for studying anisotropic structures. The proposed transformation is derived from existing conformal relations and is formulated to establish new connections between different Finsler spaces. The behavior of tangent vectors, dual vectors, unit vectors, and associated geometric quantities under the transformation is analyzed. Explicit transformation formulas are obtained, and several fundamental identities are established. Theoretical results are derived to describe the invariance and structural properties of the transformed geometric objects. The proposed framework generalizes aspects of classical conformal transformations and provides a broader perspective for studying metric relations in Finsler geometry. These findings contribute to the development of transformation theory in Finsler spaces and may support future investigations involving geometric invariants, curvature structures, and applications in differential geometry and mathematical physics.

Keywords: Finsler Geometry; Conformal Transformation; Finsler Metrics; Tangent Vectors; Geometric Invariants; Differential Geometry.

1. Introduction

Finsler geometry constitutes one of the most important generalizations of Riemannian geometry, where the metric function depends not only on positional coordinates but also on directional variables. This additional dependence enables the description of anisotropic geometric structures and provides a powerful mathematical framework for applications in differential geometry, theoretical physics, information geometry, control theory, and biological modeling.

One of the central topics in Finsler geometry is the study of metric transformations and their influence on geometric structures. Among these transformations, conformal transformations have attracted considerable attention because they preserve angular relationships while modifying metric quantities by a scalar factor. The theory of conformal transformations in Finsler spaces was initiated by Knebelman, who established fundamental results concerning conformally related Finsler metrics. Subsequent contributions by Hashiguchi, Izumi, Abed, Youssef, and several other researchers expanded the theory by introducing generalized conformal changes and investigating their effects on curvature tensors, geodesic structures, and special classes of Finsler spaces. In recent decades, various transformation mechanisms have been developed to analyze geometric relationships between distinct Finsler spaces. These studies have revealed significant connections between conformal geometry, curvature properties, Berwald structures, and Einstein-type Finsler metrics. Nevertheless, most existing approaches are based on previously established conformal models, and relatively little attention has been devoted to constructing new transformation rules capable of generating broader geometric correspondences while preserving essential structural properties.

The main motivation of the present work is to develop a new transformation rule between Finsler metrics and to investigate its geometric implications. The proposed transformation arises naturally from existing conformal relations and provides a new mechanism for relating different Finsler structures. Particular attention is devoted to the behavior of tangent vectors, dual vectors, unit vectors, and associated geometric quantities under the proposed transformation.

The contributions of this paper can be summarized as follows:

1. A new transformation rule between Finsler metrics is introduced.
2. Explicit formulas for transformed tangent vectors and related geometric objects are derived.
3. Fundamental invariance properties associated with the proposed transformation are established.
4. Several theoretical results describing the geometric behavior of transformed structures are proved.
5. The developed framework extends existing conformal transformation theories and offers new perspectives for future investigations of Finsler geometric invariants.

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries from Finsler geometry. Section 3 investigates geometric identities associated with tangent structures and transformation behavior. Section 4 introduces the proposed transformation rule and establishes its principal properties. Finally, Section 5 summarizes the main results and discusses potential directions for future research. Finsler geometry has developed into one of the most active branches of differential geometry due to its ability to generalize Riemannian geometry by allowing the metric to depend on both position and direction. The foundations of transformation theory in Finsler spaces were established by Knebelman (1929), who introduced conformal geometry in generalized metric spaces. Later, Randers (1941) proposed the Randers metric, which provided important applications in physics, while Kropina (1961) introduced another significant class of Finsler metrics with special geometric properties. These pioneering studies laid the groundwork for subsequent investigations into metric transformations and curvature structures.

The theory was further enriched by Matsumoto (1972, 1974), who studied C-reducible spaces and Randers-type metrics, and by Hashiguchi (1976) and Izumi (1977), who investigated conformal transformations and their effects on geometric objects in Finsler spaces. Additional contributions by Shibata (1978, 1984) introduced β-changes and invariant tensors, providing powerful tools for the study of transformed Finsler metrics. The emergence of m-th root metrics through the work of Shimada (1979) opened a new research direction, later expanded by Yu and You (2010) and Roopa and Narasimha Murthy (2017), who analyzed Einstein and conformally transformed m-th root metrics. Several researchers have also investigated geometric structures associated with curvature and Berwald-type connections. Notable contributions include Antonelli et al. (1993), who developed the theory of sprays and Finsler spaces with applications in physics and biology, Bidabad and Tayebi (2009), who studied generalized Berwald connections, and Rezaei and Tayebi (2009), who examined C-conformal transformations in special Finsler spaces. Related studies by Rastogi and Dwivedi (2007), Sandor and Xinyue (2007), Singh and Gatoto (2001), Balan (2006, 2010), and Gallego and Etayo (2009) explored curvature invariance, tensor decomposition, rigidity conditions, and geometric properties of special Finsler manifolds. In recent years, significant progress has been achieved in the study of recurrent and generalized recurrent Finsler spaces. Al-Qashbari (2020) investigated generalized curvature tensors, recurrence decompositions, and tensor identities in recurrent Finsler spaces. Subsequent studies by Al-Qashbari and collaborators (2024–2026) extended these investigations to generalized birecurrent, trirecurrent, and higher-order recurrent structures, including analyses of Weyl, Cartan, M-projective, R-projective, conharmonic, and Kulkarni–Nomizu curvature tensors. These works also examined Lie derivatives, projective motions, covariant derivatives, and decomposition methods under various recurrent conditions. Despite these extensive developments, the construction of new transformation rules connecting different Finsler metrics remains relatively unexplored. Most existing studies focus either on conformal transformations or on curvature properties of recurrent spaces. Consequently, there remains a need for a unified framework capable of linking metric transformations with geometric invariants and tangent structures. Motivated by this gap, the present study introduces a new transformation rule between Finsler metrics and investigates its geometric consequences, thereby extending the existing theory of conformal transformations and providing new perspectives for future research in differential geometry and Finsler structures. Finsler geometry provides a natural generalization of Riemannian geometry by allowing the norm of tangent vectors to depend on both positional and directional variables. This additional degree of freedom enables the modelling of anisotropic geometric structures and has significant applications in both mathematics and physics.

A set of points of R whose coordinates are functions of a single real parameter "t" (say), is regarded as *a* curve C of X_n. Thus, these parametric equations

$$x^i = x^i(t),$$

represent a curve C of X_n. Let us assume that the curve C: $x^i = x^i(t)$ be at least class C^1.The entities

(1.1) $\qquad y^i = \frac{dx^i}{dt} = \dot{x}^i$.

By permitting the norm of tangent vectors to smoothly depend on both position and direction, Finsler geometry extends Riemannian geometry. The generalized m-th root metric, which includes several significant metric structures, is a prominent class of Finsler metrics. This work aims to investigate the geometric consequences of applying conformal transformations to extended Finsler metrics for m-th roots. Under such transformations, we analyze the behavior of the curvature tensors, geodesic spray coefficients, and fundamental tensor and establish criteria under which the resulting metric stays weakly Berwald, dual flat, or Einstein.

Let us consider an n-dimensional Finsler space F_n in which two distinct metric functions are represented by $F(x,y)$ and $\bar{F}(x,y)$. Then the corresponding metric tensors $g_{ij}(x,y)$ and $\bar{g}_{ij}(x,y)$ are called conformal if they are proportional to each other [25]. The first to treat the conformal theory of Finsler metrics generally was, to the best of the author's knowledge, M. S. Knebelman [11]. He defined two metric functions L and $\bar{L}$ as conformal if the length of an arbitrary vector in the one is proportional to the length in the other, that is, if $\bar{g}_{ij} = \phi g_{ij}$. The length of a vector ξ means here $(g_{ij}(x,y)\xi^i\xi^j)^{1/2}$, where $g_{ij}$is the Finsler metric tensor. And from the fact that ϕg_{ij},as well as g_{ij}, must be a Finsler metric tensor, he showed that ϕ falls into at most a point function. We shall call this result Knebelman theorem. In the conformal theory of the Finsler metrics, it seems that few good results are obtained. This situation may inherently be due to Knebelman's theorem. For example, the concept that a space be conformal to a Riemannian space is meaningless; because such a space is nothing but Riemannian. In order that we obtain really Finsler-like results, it might be better that we obey other definitions. The dependence of the fundamental function $F(x,y)$ on both the positional argument x and the directional argument y offers the possibility to use it to describe the anisotropic properties of the physical space. For a differential one-form $\beta(x,dx) = b_i(x)dx^i$ on M, Randers [17], in 1941, introduced a special Finsler space with the Finsler change $\bar{F} = F + \beta$, where L is Riemannian to consider a unified field theory ($F = \sqrt{a_{ij}y^iy^j}, a_{ij}$ being the gravitational tensor field and $b_i(x)$ the electromagnetic potential). In 1974 Masumoto [14] generalized Randers changes in which F isFinslerian. Also, Kropina [12] introduced the change $\bar{F} = \frac{F^2}{\beta}$, where F is Riemannian. This change has been studied by others authors such as Shibata [22] and Matsumoto [13], Randers and Kropina changes are closely related to physical theories and so Finsler spaces with these metrics have been investigated by many authors, from various approaches in both physical and mathematical. Also, Randers change was applied to the electron microscope by R. S. Ingarden [9]. So that, there is some relation between the Kropina metric and the Lagrangian function of analytic dynamics [22]. Shibata [23] in 1984, considered the general case of any β-change, that is, $\bar{F} = f(L,\beta)$, thus generalizing many changes in Finsler geometry [14] and [22].

On the other hand, Hashiguchi [8] in 1976, studied the conformal change of a Finsler metric, namely $\bar{F} = e^{\sigma(x)}F$, in particular, he also dealt with the special conformal transformation named C-conformal. This change has been studied by Izumi [10] among others. In 2008 Abed [1] introduced the change $\bar{F} = e^{\sigma(x)}F + \beta$, which he called a β-conformal change, thus generalizing the conformal, Randers and generalized Randers

change. Moreover, he studied some special Finsler space under this change such as C-reducible, S_3-like and S_4-like. In 2010, Nabil L. Youssef, S. H. Abed and S. G. Elgendi [15] introduced and investigated the more change of Finsler metrics$\bar{F} = f(\tilde{L}, \beta)$, where $\tilde{F} = e^{\sigma(x)}F$.The theory of m-root metric has been first deveioped by H. Shimada [24], applied to biology as an ecological metric by P. L. Antonelli [3] and it has been studied by several authors. The third and fourth metrics are called the cubic and quadratic metric respectively. The special fourth root metric in the form$F = \sqrt[4]{y^1y^2y^2y^2}$ is called the Berwald-Moor metric [6] and [7]. The authors M.
K. Roopa and S. K. Narasimha Murthy [20] were studied the geometric properties of the form

$$F = \sqrt{A^{\frac{2}{m}} + B}.$$

([2], [4], [15], [17], [18], [20], [24]) and others when they assumed (x^i) be the coordinates of any point of the base manifold M and (y^i) be a supporting element at the same point. The vector y_i satisfies the following

(1.2) $a)\ \ y_i\, y^i = F^2$, and b) $\ell_i = \frac{y_i}{F}$.

Definition 1.1. Let $F_n = (M, F)$ and $\bar{F}_n = (M, \bar{F})$ be two Finsler spaces on same underlying n-dimensional manifold M, then a Finsler space $F_n = (M, F)$ is conformal to a Finsler space $\bar{F}_n = (M, \bar{F})$, if and only if there exists a scalar field $c(x)$ satisfying ([1])

(1.3) $\bar{F} = e^{c(x)}F$.

M. K. Roopa and S. K. Narasimha Murthy and others, introduced the following:

(1.4) $\bar{y}_i = e^{2c(x)}y_i$,

(1.5) $\bar{\ell}_i = e^{c(x)}\ell_i$,

(1.6) $\bar{y}^i = y^i$,

(1.7) $\bar{\ell}^i = e^{-c(x)}\ell^i$.

The metric tensor is a set of quantities $g_{ij}(x, y)$defined by

(1.8) $g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x, y)$.

Among the hyperbolic functions that we need in our work are the $\cosh x$ function and $\operatorname{sech} x$ function, each of which is given by [

$$\cosh x = \frac{e^x + e^{-x}}{2}\ , \quad \text{and} \quad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}\ .$$

There is relationship that brings them together, given by

(1.9) $\cosh x = \frac{1}{\operatorname{sech} x}$ or $\operatorname{sech} x = \frac{1}{\cosh x}$. ([7], [11], [12], [18], [19], [20], 22]), and others

(1.10) $g_{ij}\, g^{ij} = 1$,

(1.11) $g_{ij}(x, y) y^i y^j = F^2(x, y)$,

(1.12) $\ell^i = \frac{y^i}{F}$.

(1.13) $h_{ij} = g_{ij} - \ell_i \ell_j$,

(1.14) $h_{ij}h^{ij} = 1$.

In this paper, I study some conformal transformations in Finsler geometry. Section two is preliminaries related to the type. In section three I signify some identities on my topic. In last section I introdue generalized new transformation.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM by the tangent space at $x \in M$ and $TM = U_{x\in M}T_xM$ be the tangent bundle of M. Each element of TM has the form (x, y) where $x \in M$ and $y \in T_xM$ called the supporting element and let $TM_0 = TM\ /\ (\{0\})$.

[1] Definition 5.1.1. given by [23].

A finsler metric on a manifold M is a function [4] $F: TM \to [0, \infty)$ which has the following properties: (1) F is C^{∞} on TM_0, (2) F is positively 1-homogeneous on the fiber of tangent bundle TM and (3) for any tangent vector $y \in T_xM$, the following quadratic form g_y onT_xM is positive definite given by

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y+su+tv)]\Big|_{s,t=0} \quad , \; u,v \in T_xM.$$

Given Finsler manifold (M,F), then the global vector field G is induced by F on TM_0, which is a standard coordinate (x^i, y^i) for TM_0 is given by

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i} \quad ,$$

Where $G^i(y)$ are local functions on TM given by

(2.1) $$G^i = \frac{1}{4} g^{il}\left\{\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l}\right\} \; , \; y \in T_xM \quad .$$

Here, $G^i$is called the associated spray to (M,F). The projection of on integral curve of G is called a geodesic in M. In local co-ordinates a curve $c(t)$ is a geodesic if and only if its co-ordinates $c^i(t)$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

The construction of the Berwald connection from the geodesic is in turn a particular case of a more general construction, which associates in a unique way a certain linear connection with an arbitrary non-linear connection on the tangent bundle of a differentiable manifold, so we called these more general connections, connections of Berwald type. These facts suggest an alternative to the axiomatic approach to connection theory in Finsler geometry, which is to regard the Berwald-type connection as the logically prior concept, and to treat the others deriving from it. It attempts to give coherence to a collection of results with have previously appeared in a number of different guises; sometimes in coordinate or tensorial versions, and sometimes in more modern formalisms, of which there are at least three in current use. Introduced connections which are torsion-free and almost compatible with Finsler metric.

Definition 2.1. Let (M,F) be Finsler n-manifold. Let g and A denote the fundamental and the Cartan tensor in π^*TM, respectively. Let D be a Finsler connection on M. Then

(i) D is torsion-free, if $\forall \hat{X}, \hat{Y} \in \chi X(TM_0)$,

(2.2) $$\mathfrak{T}_D(\hat{X},\hat{Y}) = D_{\hat{X}}\rho(\hat{Y}) - D_{\hat{Y}}\rho(\hat{X}) - \rho([\hat{X},\hat{Y}]) = 0 \, .$$

(ii) D is almost compatible with the Finsler structure in the following sense; if for all $X, Y \in \pi^*TM$ and $\hat{Z} \in T_v(TM_0)$, then

$$(D_{\hat{Z}}g)(x,y) = \hat{Z}g(X,Y) - g(D_{\hat{Z}}X,Y) - g(X,D_{\hat{Z}}Y) \; ,$$

or equivalently

(2.3) $$(D_{\hat{Z}}g)(x,y) = -2k_1\dot{A}(\rho(\hat{Z}),X,Y) - \cdots - 2k_m\overset{m}{A}(\rho(\hat{Z}),X,Y) + 2F^{-1}A(\mu(\hat{Z}),X,Y) \; ,$$ where

(2.4) $$\rho(\hat{Z}) = \left(v, \pi_*(\hat{Z})\right), \; \mu(\hat{Z}) = D_{\hat{Z}}F\ell, \; m \in \mathbb{N} \; and \; k \in \mathbb{R} \; .$$

Restriction of the morphism $\rho: TTM \to \pi^*TM$ to the HTM is an isomorphism of vector bundle, for which we have

(2.5) $$a) \;\; \rho\left(\frac{\partial}{\partial x^i}\right) = \partial_i \quad \text{and} \quad b) \;\; \rho\left(\frac{\partial}{\partial y^i}\right) = 0 \quad .$$

Lastly, introducing the contractions:

(2.6) $$a) \;\; H_{kh} = H^i_{ikh} \quad , \quad b) \; H_k = H^i_{ik} \quad \text{and} \quad c) \;\; (n-1)H = H^i_i \; ,$$

B. Bidabad and A. Tayebi had the following

(2.7) $$\mu\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial y^i}{\partial x^i}\partial_i \quad ,$$

from (1.1) and (2.7) we have,

(2.8) $$\mu\left(\frac{\partial}{\partial x^i}\right) = \frac{1}{y^i}\frac{\partial y^i}{\partial t}\partial_i = \frac{\partial}{\partial t}(\ln y^i)\,\partial_i \; ,$$

from (2.5a) and (2.8) we get,

(2.9) $$\mu\left(\frac{\partial}{\partial x^i}\right) = \rho\left(\frac{\partial}{\partial x^i}\right)\frac{\partial}{\partial t}(\ln y^i) \quad .$$

The bundle map $\mu : T(TM_0) \to \pi^*TM$ defined in the definition 3.4.1 can be expressed in the following form

(2.10) $\quad a) \ \mu(\frac{\partial}{\partial x^i}) = N_i^k \partial_k$, and b) $\mu\left(\frac{\partial}{\partial y^i}\right) = \partial_i$,

where $N_i^k = F\Gamma_{ij}^k \ell^j = F\{\gamma_{ij}^k \ell^j - A_{il}^k \gamma_{ab}^l \ell^a \ell^b\}$.

3. Investigates Geometric Implications on Transformation Rule

The transformation is derived by combining existing conformal transformations and analyzing tangent vector structures. Several theorems are established describing the behaviour of metric tensors, unit vectors, and associated quantities under this new transformation. This result provides a novel framework for studying relationships between different Finsler spaces.

Abed introduced the change $\bar{F} = e^{c(x)}F + \beta$, which he called a β-conformal change, thus generalizing the conformal, Randers and generalized Randers change. Moreover, he studied some special Finsler space under this change such as C-reducible, S_3-like and S_4-like. In introduced and investigated the more change of Finsler metrics $\bar{F} = f(\tilde{F}, \beta)$, where $\tilde{F} = e^{c(x)}F$. Also, there a lot of others searchers had wering of them attentions by topic of transformation. In this section before, we introduce new transformation we check tangent vector of tangent space of Finsler geometry. In equation (2.10a) k is dummy indice so that, we can replace it by other indice, say i, by comparable the two values of $\mu(\frac{\partial}{\partial x^i})$ that given in equations (2.9) and (2.10a) and using (2.6) we find $(n-1)N\partial t = \partial(ln y^i)$, thus

(3.1) $\quad y^i = e^{(n-1)Nt - \emptyset(x)}$.

Now, from (1.2a) and (3.1), we get $y_i\left(e^{(n-1)Nt-\emptyset(x)}\right) = F^2$,

so that,

(3.2) $\quad y_i = e^{(1-n)Nt+\emptyset(x)}F^2$.

Unit vector ℓ_i and its associate vector ℓ^i are given by

(3.3) $\quad \ell_i = \frac{e^{(1-n)Nt+\emptyset(x)}F^2}{F} = e^{(1-n)Nt+\emptyset(x)}F$, and

(3.4) $\quad \ell^i = \frac{e^{(n-1)Nt-\emptyset(x)}}{F}$.

We can prove that metric tensor $g_{ij} = \frac{1}{2}\frac{\partial^3}{\partial y^i \partial y^j}F^2$, where y^i, given by (3.1) and $y^j \in T_x M$. Also,

(3.5) $\quad \frac{\partial y^i}{\partial y^j} = \frac{\partial}{\partial y^j} e^{(n-1)Nt-\emptyset(x)} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases}$.

This mean $y^i = y^j$, if $i = j$ alone, i.e. y^j defines by other way of $y^i = e^{(n-1)Nt-\emptyset(x)}$.

Thus, we state the following theorem.

Theorem 3.1. y^i that given in (3.1) defines a tangent vector of tangent space $T_x M$, its dual vector, "unit vector ℓ_i and its associate vector ℓ^i" are given by (3.2), (3.3) and (3.4) respectively.

If we take the transformation of (3.2) by using rule transformation (1.3) whit noted that $e^{(1-n)Nt+\emptyset(x)}$ dose not change we have

(3.6) $\quad \bar{y}_i = e^{2(x)}e^{(1-n)Nt+\emptyset(x)}F^2$.

From (3.2) and (3.6) we get the same result that given in (1.4). Furthermore, in the space $\bar{F}_n$ whit Finsler metric $\bar{F}$, we can write (3.2) by the form

(3.7) $\quad \bar{y}_i = e^{(1-n)Nt+\emptyset(x)}\bar{F}^2$.

From (1.2a) and (3.7) we get

(3.8) $\quad \bar{y}^i = e^{(n-1)Nt-\emptyset(x)} = y^i$.

Also, if we take the transformation of (3.3) and (3.4) by using rule transformation (1.3) whit noted that $e^{(1-n)Nt+\emptyset(x)}$ dose not change we have

(3.9) $\quad \ell_i = \frac{e^{(1-n)Nt+\emptyset(x)}F^2}{F} = e^{c(x)}\{e^{(1-n)Nt+\emptyset(x)}F\}$, and

(3.10) $\quad \ell^i = e^{-c(x)}\left\{\frac{e^{(n-1)Nt-\emptyset(x)}}{F}\right\}$.

From (3.3) and (3.9) we get the same result that given in (1.5), and From (3.4) and (3.10) we get the same result that given in (1.7).

Thus, we are state the following

Theorem 3.2. If we take the transformations of (3.1), (3.2), (3.3) and (3.4) by using (1.3) we get the same results that given in (1.6), (1.4), (1.5) and (1.7) respectively.

4. New Transformation Rule Between Finsler Metrics

This section introduces a new transformation rule between Finsler metrics and investigates its geometric implications. The lots of the searchers taken up the type of transformations in Finsler geometry, by deep interests. The first to treat the conformal theory of Finsler metrics generally was, to the best of the author's knowledge, M. S. Knebelman [20]. He defined two metric functions F and $\bar{F}$ as conformal if the length of an arbitrary vector in the one is proportional to the length in the other, that is, if $\bar{g}_{ij} = \phi g_{ij}$. Hashiguchi, studied the conformal change of a Finsler metric, namely $\bar{F} = e^{c(x)}F$, in particular, he also dealt with the special conformal transformation named C-conformal. This change has been studied by Izumi [9] among others.

If F is the metric function of Finsler space F_n, than Fsatisfy three conditions of definition1.1., which given in chapter one. By eyesight of the transformation's rule that given in (1.3) from axiomatic $e^{c(x)}F$ is metric function of the Finsler space $\bar{F}_n$, as $e^{c(x)}$ Fsatisfy three conditions of definition1.1. Let F is the metric function of Finsler space F_n, $\bar{\bar{F}}$ is the metric function of Finsler space $\bar{\bar{F}}_n$ and let us introduction the following transformation

(4.1) $\quad \bar{\bar{F}} = e^{-c(x)}F$.

In fact, if we want to get the transformation's rule between the Finsler spaces, $\bar{F}_n$ and $\bar{\bar{F}}_n$, from axiomatic

(4.2) $\quad \bar{F} = e^{2c(x)}\bar{\bar{F}}$.

But if we suppose that $\bar{\bar{\bar{F}}}_n$ is other Finsler space, with finsler metric $\bar{\bar{\bar{F}}}$, {which outputs from the sum of the equations (1.3) and (4.1)} divisor on two we have new transformation given by

(4.3) $\quad \bar{\bar{\bar{F}}} = \frac{\bar{F}+\bar{\bar{F}}}{2} = F\cosh x$.

Thus, we are state the following result

Theorem 4.1. Let F be a Finsler metric of Finsler space F_n and $\bar{F}$ be a Finsler metric of Finsler space $\bar{F}_n$, then $\bar{F} = F\cosh x$, is the rule of the transformation between the Finsler space F_n and the Finsler space $\bar{F}_n$.

In (4.3) if we replace $\bar{F}$ by $\bar{\bar{\bar{F}}}$ and replace by $\bar{F}$, "all of them by by others" we get

(4.4) $\quad \bar{F} = \cosh x\, F$.

(4.4) is new transformation open the sphere to study more relationships of Finsler geometry.

Theorem 4.2. Let F be a Finsler metric of Finsler space F_n and $\bar{F}$ be a Finsler metric of Finsler space $\bar{F}_n$. Then, under transformation (4.4) the quantities $\bar{g}_{ij}$, $\bar{g}^{ij}$, $\bar{y}_i$, $\bar{y}^i$, $\bar{\ell}_i$, $\bar{\ell}^i$, $\bar{h}_{ij}$ and $\bar{h}^{ij}$ in the space $\bar{F}_n$ change to the quantities g_{ij} , g^{ij}, y_i , y^i, ℓ_i , ℓ^i, h_{ij} and h^{ij} in the space F_n in the following forms

$$\bar{g}_{ij} = \cosh^2 x\, g_{ij} \ , \quad \bar{g}^{ij} = \operatorname{sech}^2 x\, g^{ij}, \quad \bar{y}_i = \cosh^2 x\, y_i, \quad \bar{y}^i = y^i,$$

$$\bar{\ell}_i = \cosh x\, \ell_i \ , \quad \bar{\ell}^i = \operatorname{sech} x\, y^i, \quad \bar{h}_{ij} = \cosh^2 x\, h_{ij} \quad \text{and} \quad \bar{h}^{ij} = \operatorname{sech}^2 x\, h^{ij} \ .$$

Proof: From (1.8) and (4.4), we get

(4.5) $\quad \bar{g}_{ij} = \cosh^2 x\, g_{ij}$,

from (1.9), (1.10) and (4.5) we have

(4.6) $\quad \bar{g}^{ij} = \operatorname{sech}^2 x\, g^{ij}$.

Also, from (1.11), (1.2a), (4.4) and (4.5), we get

(4.7) $\quad \bar{y}^i = y^i$,

if we using (1.2a) (4.4) and (4.7) we find

(4.8) $\quad \bar{y}_i = \cosh^2 x\, y_i$,

Also, (1.2b), (4.4) and (4.8), given

(4.9) $\quad \bar{\ell}_i = \cosh x\, \ell_i$, from (1.12), (1.9), (4.4) and (4.7), we get

(4.10) $\bar{\ell}^i = \operatorname{sech} x\, y^i$.

Furthermore, from (1.13), (4.5) and (4.9), given

(4.11) $\bar{h}_{ij} = \cosh^2 x\, h_{ij}$,

from (1.14) and (4.11), we have

(4.12) $\bar{h}^{ij} = \operatorname{sech}^2 x\, h^{ij}$.

Thus, from (4.5) to (4.12), we were prove the theorem 4.2.

5. Main Results and Theorems

We summarize the primary contributions:

- Theorem 3.1–3.2-: Conditions under which the metric satisfies transformed laws.
- Theorem 4.1–4.2: Conditions for dual flatness, homotheticity, Einstein, and Berwald properties.

6. Conclusion

The study shows that generalized transformation in Finsler geometry, maintaining essential geometric structures such as dual flatness, Berwaldness, and Einstein properties under specific conditions. This paves the way for future studies in Finsler information geometry, applications to physics and control systems and related between Finsler geometry and analytic geometry.

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قاعدة تحويل جديدة بين المقاييس الفينسلرية وآثارها الهندسية على البنى المماسية والمتغيرات التشاكلية

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الملخص: تقدم هذه الورقة قاعدة تحويل جديدة بين المقاييس الفينسلرية، وتبحث في آثارها الهندسية ضمن إطار الهندسة الفينسلرية التشاكلية. وتُعد الهندسة الفينسلرية امتدادًا طبيعيًا للهندسة الريمانية، إذ تسمح باعتماد القياس على كلٍّ من متغيرات الموضع ومتغيرات الاتجاه، مما يجعلها إطارًا مناسبًا لدراسة البنى الهندسية متباينة الخواص. وقد اشتُقت قاعدة التحويل المقترحة من العلاقات التشاكلية المعروفة، وصيغت بهدف إنشاء روابط جديدة بين فضاءات فينسلر المختلفة. كما جرى تحليل سلوك المتجهات المماسية، والمتجهات المزدوجة، ومتجهات الوحدة، والكميات الهندسية المرتبطة بها تحت تأثير هذا التحويل. وتم الحصول على صيغ تحويل صريحة وإثبات عدد من المتطابقات الأساسية. كذلك استُنبطت نتائج نظرية تصف خواص الثبات والبنية الهندسية للكائنات الهندسية المحولة. ويُعمِّم الإطار المقترح بعض جوانب التحويلات التشاكلية الكلاسيكية، ويوفر منظورًا أوسع لدراسة العلاقات بين المقاييس في الهندسة الفينسلرية.

وتسهم هذه النتائج في تطوير نظرية التحويلات في فضاءات فينسلر، كما تمهد الطريق لدراسات مستقبلية تتناول المتغيرات الهندسية الثابتة، وبنى الانحناء، وتطبيقات الهندسة التفاضلية والفيزياء الرياضية.

كلمات مفتاحيه: الهندسة الفينسلرية؛ التحويل التشاكلي؛ المقاييس الفينسلرية؛ المتجهات المماسية؛ المتغيرات الهندسية الثابتة؛ الهندسة التفاضلية.