A New Class of Generalized U-Birecurrent Finsler Spaces and Their Curvature Properties

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Abstract: In this paper, we introduce a new class of generalized U-birecurrent Finsler spaces defined in terms of second-order Berwald covariant derivatives of the h(v)-curvature tensor. This new structure extends existing concepts of recurrent and birecurrent Finsler spaces by incorporating higher-order differential conditions. Several fundamental properties of the proposed spaces are established, including relationships between the h(v)-curvature tensor and other important curvature tensors such as Cartan's curvature tensors, Berwald curvature tensor, and the normal projective curvature tensor. Furthermore, a series of theorems and corollaries are derived to characterize the geometric behavior of these spaces under generalized recurrence conditions. The obtained results provide a broader framework for the study of curvature structures in Finsler geometry and contribute to the ongoing development of higher-order geometric analysis.

Key words: Finsler geometry; generalized birecurrent space; U-birecurrence; Berwald covariant derivative; curvature tensor; h(v)-curvature tensor; Cartan curvature tensor; projective curvature tensor.

1. Introduction

Finsler geometry is regarded as a natural and far-reaching generalization of Riemannian geometry, in which the metric depends on both position and direction. This additional degree of freedom allows Finsler spaces to model a wide range of anisotropic phenomena arising in mathematics and physics. In recent years, significant progress has been made in the study of curvature structures and tensorial invariants in Finsler manifolds, particularly in relation to geometric analysis and modern physical applications [32–36].

Curvature tensors play a central role in understanding the intrinsic geometric properties of Finsler spaces. Various types of curvature tensors, including Cartan, Berwald, and projective curvature tensors, have been extensively studied to describe geometric behavior under different connections and transformations. Recent works have emphasized the importance of higher-order curvature tensors and their associated invariants in developing a deeper understanding of Finsler structures [33–36]. Moreover, extensions of curvature tensors from general relativity and relativistic spacetime geometry have also been investigated, highlighting connections between Finsler geometry and theoretical physics [1–3]. A major direction of research in Finsler geometry is the study of recurrent and generalized recurrent structures. These structures describe spaces in which curvature tensors satisfy specific differential conditions, leading to important geometric and symmetry properties. Several types of recurrence, including recurrent, birecurrent, and trirecurrent Finsler spaces, have been introduced and analyzed using different covariant derivatives such as those defined by Cartan and Berwald connections [4-6, 15]. In particular, generalized recurrence decompositions and tensor identities have been studied to understand the behavior of curvature tensors under various geometric constraints [4-6].

More recently, attention has shifted toward higher-order generalizations of recurrence conditions and their applications. A number of studies have introduced new classes of generalized recurrent Finsler spaces using higher-order covariant derivatives, Lie derivatives, and mixed derivative structures [7-12, 18]. These approaches have led to significant developments in understanding the interaction between different curvature tensors, including Weyl, projective, and conharmonic curvature tensors, within generalized geometric frameworks [10-12, 16]. Furthermore, decomposition techniques and tensorial analysis have been applied to investigate structural properties of Finsler spaces and their geometric invariants [11, 16].

In addition, recent contributions have explored advanced topics such as generalized trirecurrence, higher-order curvature decompositions, and applications of Finsler geometry in areas such as geometric modeling, mechanics, and computational methods [13, 14, 17, 19, 20]. These developments indicate a growing interest in extending classical recurrence concepts to more general and flexible geometric settings.

Despite these advances, most existing studies are primarily limited to first-order recurrence conditions or specific curvature tensors. The role of higher-order Berwald covariant derivatives in defining generalized birecurrent structures, particularly in relation to the h(v)-curvature tensor, remains insufficiently explored. This gap motivates the need for developing a more comprehensive framework that incorporates higher-order differential structures into generalized recurrence theory.

In this paper, we introduce a new class of generalized U-birecurrent Finsler spaces defined via second-order Berwald covariant derivatives of the h(v)-curvature tensor. This formulation provides a natural extension of existing recurrence structures and allows for a deeper analysis of higher-order curvature behavior. We establish several fundamental properties of these spaces and derive new relationships between the h(v)-curvature tensor and other important curvature tensors, including Cartan, Berwald, and normal projective curvature tensors.

The results obtained in this work generalize several known concepts in Finsler geometry and contribute to the ongoing development of higher-order geometric analysis. Moreover, they provide a unified framework for studying curvature interactions in generalized Finsler spaces and open new directions for future research in both theoretical and applied contexts.

Let F_n be an n-dimensional Finsler space equipped with the metric function $F(x, y)$ satisfying the request conditions

Let us consider a set of quantities $g_{ij}(x, y)$ defined by

(1.1) $\quad g_{ij}(x\,, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x\,, y)$.

Where g_{ij} is positively homogeneous of degree zero in directional arguments.

(1.2) $\quad y_i = g_{ij}y^j$.

The quantities g_{ij} and g^{jk} are related by

(1.3) $\quad g_{ij}g^{jk} = \delta_i^k = \begin{cases} 1 & if \quad i = k, \\ 0 & if \quad i \neq k. \end{cases}$

The vectors y_i and y_k, related with g_{ik} and g^{jk} by

(1.4) $\quad$ a) $\delta_k^i\, y_i = y_k$, b) $\delta_k^i y^k = y^i$, c) $\delta_j^i\, g_{ik} = g_{jk}$, d) $\dot{\partial}_j\, y^i = \delta_j^i$,

e) $\delta_j^i g^{jk} = g^{ik}$, f) $\delta_j^i\, \delta_k^j = \delta_k^i$, and g) $\delta_i^i = n$.

(1.5) $\quad C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2$,

where C_{ijk} is known as (h)hv-torsion tensor, it is positively homogeneous of degree -1 in y^i and symmetric in all its indices.

By using Euler's theorem on homogeneous properties, this tensor satisfies the following identities

(1.6) $\quad$ a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$, b) $C_{jk}^i y^j = C_{kj}^i\, y^j = 0$,

c) $C_{jk}^i y_i = 0$, and d) $\delta_j^r C_{rkh} = C_{jkh}$.

L. Berwald defined covariant derivative for his connection parameter G_{jk}^i. Thus, Berwald's covariant derivative $\mathcal{B}_k\, X^i$ of an arbitrary vector field X^i with respect to x^k is given by

(1.7) $\quad \mathcal{B}_k\, X^i = \partial_h\, X^i - (\dot{\partial}_r X^i)\, G_k^i + X^r G_{rk}^i$.

Similar to Cartan's covariant derivative, Berwald's covariant derivative of the metric function F, the unit $l^i$and the vector y^i vanish identical, *i.e.*

(1.8) $\quad$ a) $\mathcal{B}_k F = 0$, b) $\mathcal{B}_k l^i = 0$, and c) $\mathcal{B}_k\, y^i = 0$.

The covariant derivative of the metric tensor g_{ij} does n't vanish in sense of Berwald, i.e. $\mathcal{B}_k g_{ij} \neq 0$.

The covariant derivative of the metric tensor is given by

(1.9) $\mathcal{B}_k g_{ij} = -2C_{ijk|h} y^h = -2y^h \mathcal{B}_h C_{ijk}$.

The processes of Berwald`s covariant differentiation and the partial differentiation commute according to

(1.10) $(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k)\, T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}$, for an arbitrary tensor field T^i_j .

The tensor $K^{\,i}_{rkh}$ called Cartan's fourth curvature tensor which is skew-symmetric in its last two lower indices k and h, $i.e.$

(1.11) $K^{\,i}_{jkh} = -K^{\,i}_{jhk}$

The associate curvature tensor K_{ijkh} of the curvature tensor $K^{\,i}_{jkh}$ is given by

(1.12) $K_{ijkh} = g_{rj}\, K^r_{ikh}$.

The associate curvature tensor K_{ijkh} of the curvature tensor $K^{\,i}_{jkh}$ is skew-symmetric in its first two lower indices k and h, $i.e.$

(1.13) $K_{ijhk} = -K_{jihk}$.

The K-Ricci tensor K_{jk} , the curvature scalar K and the deviation tensor K^i_j of the curvature tensor K^i_{jkh} are satisfying the following

(1.14) $a)$ $K^i_{jki} = K_{jk}$, b) $g^{jk} K_{jk} = K$, c) $g^{ik} K_{jk} = K^i_j$, and d) $K^i_k\, g_{jh} = K^i_{jkh}$.

The curvature tensor $K^{\,i}_{jkh}$ satisfies the following relations too

(1.15) $K^i_{jkh}\, y^j = H^i_{kh}$,

(1.16) $H^i_{jkh} = K^i_{jkh} + y^s(\dot{\partial}_j\, K^i_{skh})$

The tensor R^i_{jkh} called h-curvature tensor (Cartan's third curvature tensor), this tensor is positively homogeneous of degree –1 in y^i and skew-symmetric in its last two lower indices k and h, i.e.

(1.17) $R^i_{jkh} = -R^i_{jhk}$, and satisfies the relations

(1.18) $R^i_{jkh} = K^i_{jkh} + C^i_{jm}\, H^m_{kh}$, and

(1.19) $R^i_{jkh} y^j = H^i_{kh}$.

The R-Ricci tensor R_{jk}, the deviation tensor R^r_h and curvature scalar R of the curvature tensor R^i_{jkh} are given by

(1.20) a) $R^i_{jki} = R_{jk}$, b) $R^r_{ikh}\, g^{ik} = R^r_h$, and c) $g^{jk} R_{jk} = R$.

In contracting the indices i and h in the equation (1.18), we get

(1.21) $R_{jk} = K_{jk} + C^r_{jm}\, H^m_{kr}$.

Berwald's tensors H^i_{jkh} is skew-symmetric in the last two lower indices, i.e. k and h and positively homogeneous of degree zero. i.e

(1.22) $H^i_{jkh} = -H^i_{jhk}$.

Torsion tensor H^i_{kh} of Berwald curvature tensor H^i_{jkh} is given by

(1.23) a) $H^i_{jkh}\, y^j = H^i_{kh}$ and b) $H^i_{jkh} = H_{jk}$.

The tensor U^i_{jkh} denoted the tensor $\prod^i_{jkh}$ by U^i_{jkh} and defined by

(1.24) $U^i_{jkh} = G^i_{jkh} + \frac{1}{(n+1)}\left(\delta^i_j G^r_{khr} + y^i G^r_{jkhr}\right)$.

Also, the tensor U^i_{jkh} is called h(v)-curvature tensor and it is homogeneous of degree -1in y^i and symmetric in its last two indices and satisfies the following

(1.25) a) $U^i_{jkh} = U^i_{jhk}$, b) $U^h_{jkh} = U^h_{jhk}$, c) $U^i_{jkh} = U^i_{jhk}$,
d) $U^i_{jkh}\, y^h = U^i_{jhk}\, y^h = U^i_{jk}$, and e) $U^i_{jkh}\, y^j = 0$.

The h(v)-Ricci tensor U_{jk} of the projective connection coefficients satisfies the following:

(1.26) a) $U^i_{ikh} = U_{kh}$, b) $U^h_{jkh} = U_{jk}$, where c) $U_{kh} = \frac{2}{(n+1)} G_{kh}$,

where the tensor G_{jk} is components of the call h(v)-Ricci tensor.
The h(v)-torsion tensor U^i_{jk} satisfies
(1.27) a) $U^i_{jk} = U^i_{kj}$, b) $U^i_{jk} = U^i_{jkh}\, y^h$, and c) $U^r_{jr} = C^r_{jr}$ and d) $U^r_{jkr} = C^r_{jkr}$.
P.N. Pandey obtained a relation between the normal projective curvature tensor N^i_{jkh} and Berwald curvature tensor H^i_{jkh} as follows:
(1.28) $N^i_{jkh} = H^i_{jkh} - \frac{1}{(n+1)}\, y^i\, \dot{\partial}_j H^r_{rkh}$.
The normal projective curvature tensor N^i_{jkh} is skew-symmetric in its last two lower indices, i.e.
(1.29) $N^i_{jkh} = -N^i_{jhk}$, and
(1.30) $N_{jk} = N^r_{jkr}$.
U-recurrent Finsler space, U^h-birecurrent Finsler space and U-trirecurrent Finsler space are characterized by
(1.30) a) $\mathcal{B}_l U^i_{jkh} = \lambda_l U^i_{ikh}$, where $U^i_{ikh} \neq 0$, and
b) $\mathcal{B}_m \mathcal{B}_l U^i_{jkh} = a_{lm}\, U^i_{ikh}$.
A Finsler space which the h(v)-curvature tensor U^i_{ikh} satisfies the general $U_{|l}$-recurrent Finsler space and general $U_{|l}$-birecurrent Finsler space characterized by the following conditions, respectively [10]
(1.31) a) $\mathcal{B}_l U^i_{jkh} = \lambda_l U^i_{jkh} + b\mu_l(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$, $U^i_{jkh} \neq 0$,
(1.32) b) $\mathcal{B}_m \mathcal{B}_l U^i_{jkh} = a_{lm} U^i_{jkh} + b_{lm}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) - 2\mu_l y^r \mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h C_{jkm})$, where $w_{lm} = \lambda_{l|m} + \lambda_l \lambda_m$ and $v_{lm} = \lambda_l \mu_m + \mu_{l|m}$ are non-zero covariant curvature tensors fields of second order.
(1.34) $\mathcal{B}_m \mathcal{B}_l U^i_{jkh} = w_{lm} U^i_{jkh} + v_{lm}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) - 2\mu_l y^r \mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h C_{jkm})$, where $\mathcal{B}_m \mathcal{B}_l$ is Berwald's covariant differential operator of the second order with respect to x^l and x^m respectively, $a_{lm} = \mathcal{B}_m \lambda_l + \lambda_l \lambda_m$ and $b_{lm} = \mathcal{B}_m \mu_l + \lambda_l \mu_m$ are non-zero covariant tensors field and μ_l is non-zero covariant vector field of first order.

2. An Extended Framework for Generalized $\mathcal{B}_h$U-Birecurrent Finsler Spaces

An advanced of generalized birecurrent structures in Finsler spaces related to the h(v)-curvature tensor is shown in this section. In the context of Berwald's connection, the goal of this section is to develop a generalized recurrence condition incorporating higher-order covariant derivatives.
Let F_n be an n-dimensional Finsler space with the Cartan connection. Let U^i_{jkh} denote be the space's h(v)-curvature tensor.
Assume that the generalized recurrent condition is satisfied by the h(v)-curvature tensor the following condition:
(2.1) $\mathcal{B}_l U^i_{jkh} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) + \frac{1}{4}\eta_l(U^i_k g_{jh} - U^i_h g_{jk})$, $U^i_{jkh} \neq 0$,
Where U^i_j is a deviation tensor connected to the curvature tensor U^i_{jkh}, and λ_l, μ_l and η_l are non-zero covariant vectors fields of first order.
The tensor U^i_{jkh} will be referred to as a generalized recurrent tensor, and a Finsler space that satisfies condition (2.1) will be called a generalized recurrent Finsler space. We will refer to the tensor as the generalization generalized $\mathcal{B}_h U$-recurrent tensor and. We will abbreviate them as $\mathrm{G}^{2\mathrm{nd}}\mathcal{B}_h U\text{-}RF_n$ and $\mathrm{G}^{2\mathrm{nd}}\mathcal{B}_h U\text{-}R$, respectively.
Remark. 2.1. If we substitute any other deviation tensors, such as K^i_k and K^i_h or R^i_k and R^i_h or H^i_k and H^i_h , or N^i_k and N^i_h , for U^i_k and U^i_h , we obtain a new condition that characterizes a generalized generic $\mathcal{B}_h U$-recurrent Finsler space in condition (2.1).
Taking $\mathcal{B}$-covariant derivative for (2.1) with respect to x^m, in the since of Berwald and using (1.8) we get
(2.2) $\mathcal{B}_m \mathcal{B}_l U^i_{jkh} = (\mathcal{B}_m \lambda_l) U^i_{jkh} + \lambda_l(\mathcal{B}_m\, U^i_{jkh}) + (\mathcal{B}_m \mu_l)\,(\delta^i_j\, g_{kh} - \delta^i_k\, g_{jh})$
$-2\,\mu_l\, y^r \mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h\, C_{jkm}) + \frac{1}{4}(\mathcal{B}_m \eta_l)\,(U^i_k\, g_{jh} - U^i_h\, g_{jk})$

$$+\tfrac{1}{4}\eta_l[(\mathcal{B}_m U^i_k)g_{jh}-(\mathcal{B}_m U^i_h)g_{jk}]-\tfrac{1}{2}\eta_l\, y^r(U^i_k\,\mathcal{B}_r C_{jhm}-U^i_h\,\mathcal{B}_r C_{jkm}) \ .$$

Using (2.1) in (2.2), we get

(2.3) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh}=w_{lm}U^i_{jkh}+v_{lm}(\delta^i_k g_{jh}-\delta^i_h g_{jk})+\tfrac{1}{4}c_{lm}(U^i_k\, g_{jh}-U^i_h\, g_{jk})$$
$$-2\,\mu_l\, y^r\mathcal{B}_r(\delta^i_k\, C_{jhm}-\delta^i_h\, C_{jkm})-\tfrac{1}{2}\eta_l y^r(U^i_k\,\mathcal{B}_r\, C_{jhm}-U^i_h\,\mathcal{B}_r C_{jkm})$$
$$+\tfrac{1}{4}\eta_l\,[(\mathcal{B}_m U^i_k)g_{jh}-(\mathcal{B}_m U^i_h)g_{jk}]\ .$$

Where $w_{lm}=\mathcal{B}_m\lambda_l+\lambda_l\lambda_m$, $v_{lm}=\lambda_l\mu_m+\mathcal{B}_m\mu_l$ and $c_{lm}=\lambda_l\eta_m+\mathcal{B}_m\eta_l$ are non-zero covariant tensors field of second order and η_l and μ_l are non- zero covariant vectors fields.

Definition 2.1. A Finsler space F_n for which h(v)-curvature tensor U^i_{jkh} satisfies the condition (2.3), where w_{lm}, v_{lm} and c_{lm} are non -zero covariant tensors fields of second order, and η_l and μ_l are non- zero covariant vectors fields of first order, is called *generalization generalized* $\mathcal{B}_h U$-birecurrent Finsler Space and the tensor will be called the *generalization* generalized $\mathcal{B}U$-recurrent tensor, we shall denoted them briefly by $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$ and $G^{2nd}\,\mathcal{B}U\text{-}BRF_n$ respectively.

Result 2.1. All $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$ is $G^{2nd}\,\mathcal{B}U\text{-}BRF_n$.

Transvecting, (2.3) by y^h and using (1.15d), (1.8c), (1.2), (1.4b), (1.6a) and (1.6d), we get

(2.4) $$\mathcal{B}_m\mathcal{B}_l U^i_{jk}=w_{lm}U^i_{jk}+v_{lm}(\delta^i_k\, y_j-y^i\, g_{jk})+\tfrac{1}{4}c_{lm}(U^i_k\, y_j-U^i_h\, y^h\, g_{jk})$$
$$+2\,\mu_l\, y^r\mathcal{B}_r(y^i\, C_{jkm})+\tfrac{1}{2}\,\eta_l y^r(U^i_h\, y^h\,\mathcal{B}_r\, C_{jkm})+\tfrac{1}{4}\eta_l\,[(\mathcal{B}_m\, U^i_k)y_j-(\mathcal{B}_m U^i_h\, y^h)g_{jk}]\ .$$

Suppose that $U^r_h\, y^h=U^r$ in (2.4) and using (1.8c), we get

(2.5) $$\mathcal{B}_m\mathcal{B}_l U^i_{jk}=w_{lm}U^i_{jk}+v_{lm}(\delta^i_k\, y_j-y^i g_{jk})+\tfrac{1}{4}c_{lm}(U^i_k\, y_j-U^i g_{jk})$$
$$+2\,\mu_l\; y^r y^i\mathcal{B}_r\, C_{jkm}+\tfrac{1}{2}\,\eta_l\, y^r U^i\,\mathcal{B}_r\, C_{jkm}+\tfrac{1}{4}\eta_l[(\mathcal{B}_m U^i_k)y_j-(\mathcal{B}_m U^i)g_{jk}]$$

Contracting i and h in (2.3) and using (1.4g), (1.4c), (1.6a) and (1.17b), we get

(2.6) $$\mathcal{B}_m\mathcal{B}_l U_{jk}=a_{lm}U_{jk}+(1-n)b_{lm}g_{jk}+\tfrac{1}{4}c_{lm}(U^s_k\, g_{js}-U^s_s\, g_{jk})-2\,\mu_l\, y^r\mathcal{B}_r(\delta^s_k\, C_{jsm}-\delta^s_s\, C_{jkm})$$
$$-\tfrac{1}{2}\,\eta_l y^r(U^s_k\,\mathcal{B}_r\, C_{jsm}-U^s_s\mathcal{B}_r\, C_{jkm})+\tfrac{1}{4}\eta_l\,[(\mathcal{B}_m U^s_k)g_{js}-(\mathcal{B}_m U^s_s)g_{jk}]\ .$$

Suppose that $U^r_r=U$, $U^r_k\, g_{jr}=U_{jk}$ and $\delta^r_k\, C_{jrm}=C_{jkm}$ in the above equation, we get

(2.7) $$\mathcal{B}_m\mathcal{B}_l U_{jk}=a_{lm}U_{jk}+(1-n)(b_{lm}g_{jk}-2\,\mu_l\, y^r\mathcal{B}_r\, C_{jkm})+\tfrac{1}{4}c_{lm}(U_{jK}-U g_{jk})$$
$$-\tfrac{1}{2}\,\eta_l\, y^r(U^s_k\,\mathcal{B}_r C_{jsm}-U\mathcal{B}_r C_{jkm})+\tfrac{1}{4}\eta_l[(\mathcal{B}_m U^s_k)g_{js}-(\mathcal{B}_m U)g_{jk}]\ .$$

Thus, we may conclude

Theorem. 2.1. In $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$ then, the h(v)-torsion tensor U^i_{jk} and the Ricci tensor U_{jk} are given by (3.5) and (3.7), respectively.

Contracting i and k in (2.4), using (1.4c), (1.4a), (1.8c), (1.6a) and (1.17c), we get

(2.8) $$\mathcal{B}_m\mathcal{B}_l C^s_{js}=w_{lm}C^s_{jk}+(n-1)y_j v_{lm}+\tfrac{1}{4}c_{lm}(U^s_s y_j-U^s g_{js})+2\,\mu_l\, y^r\mathcal{B}_r(0)$$
$$+\tfrac{1}{2}\,\eta_l\, y^r(U^i\,\mathcal{B}_r\, C_{jkm})+\tfrac{1}{4}\eta_l\,[(\mathcal{B}_m U^s_s)y_j-(\mathcal{B}_m U^s)g_{js}]\ .$$

Suppose that $U^s_s=U$ and $U^s g_{js}=U_j$, we get

(2.9) $$\mathcal{B}_m\mathcal{B}_l C^s_{js}=w_{lm}\, C^s_{jk}+(n-1)y_j v_{lm}+\tfrac{1}{4}c_{lm}(U y_j-U_j)$$
$$+\tfrac{1}{2}\eta_l\, y^r(U^s\,\mathcal{B}_r\, C_{jsm})+\tfrac{1}{4}\eta_l[(\mathcal{B}_m\, U)y_j-(\mathcal{B}_m U^s)g_{js}]\ .$$

Thus, we may conclude

Theorem. 2.2. In $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$, the curvature tensor C^S_{jS} satisfies (2.9).

3. Some Relationship among Curvatures Tensors In $G^{2nd}\mathcal{B}_h$U-BRF$_n$

Several relations between various curvature tensors in generalized birecurrent Finsler spaces are established in this section. Specifically, we examine how the Berwald curvature tensor, Cartan curvature tensors, and normal projective curvature tensor interacts.

By using Remark (2.1) in (2.1) and instead of U_k^i and U_h^i by the deviation tensor K_k^i and K_h^i , we get

(3.1) $\mathcal{B}_l U_{jkh}^i = \lambda_l U_{jkh}^i + \mu_l(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk}) + \frac{1}{4}\eta_l(U_k^i g_{jh} - U_h^i g_{jk})$, $U_{jkh}^i \neq 0$.

Taking $\mathcal{B}$-covariant derivative for (3.1) with respect to x^m, in the since of Berwald and using (1.9), we get

(3.2) $$\mathcal{B}_m(\mathcal{B}_l U_{jkh}^i) = (\mathcal{B}_m\lambda_l)U_{jkh}^i + \lambda_l(\mathcal{B}_m U_{jkh}^i) + (\mathcal{B}_m\mu_l)\,(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk})$$
$$- 2\,\mu_l\, y^r\mathcal{B}_r(\delta_k^i\, C_{jhm} - \delta_h^i\, C_{jkm}) + \frac{1}{4}(\mathcal{B}_m\eta_l)\,(K_k^i\, g_{jh} - K_h^i\, g_{jk})$$
$$+\frac{1}{4}\eta_l[(\mathcal{B}_m\, K_k^i)g_{jh} - (\mathcal{B}_m K_h^i)g_{jk}] - \frac{1}{2}\eta_l\, y^r\mathcal{B}_r\,[K_k^i\,(\mathcal{B}_r C_{jhm}) - K_h^i\,(\mathcal{B}_r C_{jkm})]\,.$$

Using (3.1) in (3.2), we get

(3.3) $$\mathcal{B}_m\mathcal{B}_l U_{jkh}^i = w_{lm}U_{jkh}^i + v_{lm}(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk}) + \frac{1}{4}c_{lm}(K_k^i\, g_{jh} - K_h^i\, g_{jk})$$
$$- 2\,\mu_l\, y^r\,\mathcal{B}_r(\delta_k^i\, C_{jhm} - \delta_h^i\, C_{jkm}) + \frac{1}{4}\eta_l\,[(\mathcal{B}_m\, K_k^i)g_{jh} - (\mathcal{B}_m K_h^i)g_{jk}]$$
$$- \frac{1}{2}\eta_l\, y^r[K_k^i\,(\mathcal{B}_r\, C_{jhm}) - K_h^i\,(\mathcal{B}_r C_{jkm})]\ .$$

Where $w_{lm} = \mathcal{B}_m\lambda_l + \lambda_l\,\lambda_m$, $v_{lm} = \lambda_l\,\mu_m + \mathcal{B}_m\mu_l$ and $c_{lm} = \lambda_l\eta_l + \mathcal{B}_m\eta_l$ are non-zero covariant tensors fields of second order, respectively.

Thus, we conclude

Theorem.3.3. A $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$,may characterized by the condition (3.3).

By using (1.11) and (1.14d)) in (3.3), we get

(3.5) $$\mathcal{B}_m\mathcal{B}_l U_{jkh}^i = w_{lm}U_{jkh}^i + \frac{1}{2}c_{lm}K_{jkh}^i + v_{lm}(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk}) + \frac{1}{4}\,\eta_l\,[(\mathcal{B}_m\, K_k^i)g_{jh}$$
$$-(\mathcal{B}_m K_h^i)g_{jk}] - 2\,\mu_l\, y^r\mathcal{B}_r(\delta_k^i C_{jhm} - \delta_h^i C_{jkm}) - \frac{1}{2}\eta_l y^r(K_k^i\,\mathcal{B}_r C_{jhm} - K_h^i\,\mathcal{B}_r C_{jkm})\,.$$

Thus, we may conclude

Corollary 3.1. In $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$, the relationship between the h(v)-curvature tensor U_{jkh}^i and Cartan's fourth curvature tensor K_{jkh}^i is given by (3.5).

By using (1.16) in (3.3), we get

(3.6) $$\mathcal{B}_m\mathcal{B}_l U_{jkh}^i = w_{lm}\, U_{jkh}^i + \frac{1}{4}c_{lm}H_{jkh}^i - \frac{1}{4}c_{lm}\, y^s(\dot{\partial}_j K_{skh}^i) + v_{lm}(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk})$$
$$+\frac{1}{4}\,\eta_l\,[(\mathcal{B}_m\, K_k^i)g_{jh} - (\mathcal{B}_m K_h^i)g_{jk}] - 2\,\mu_l\, y^r\mathcal{B}_r(\delta_k^i\, C_{jhm} - \delta_h^i\, C_{jkm})$$
$$-\frac{1}{2}\eta_l\, y^r[\,K_k^i\,(\mathcal{B}_r\, C_{jhm}) - K_h^i\,(\mathcal{B}_r C_{jkm})]\ .$$

Thus, we may conclude

Corollary 3.2. In $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$, the relationship between n the h(v)-curvature tensor U_{jkh}^i and Berwald's curvature tensor H_{jkh}^i given by (3.6).

Let's look at a Finsler space F_n where the h(v) curvature tensor U_{jkh}^i meet the requirements listed below:

(3.7) $\mathcal{B}_l U_{jkh}^i = \lambda_l U_{jkh}^i + \mu_l(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk}) + \frac{1}{4}\eta_l(H_k^i g_{jh} - H_h^i g_{jk})$, $U_{jkh}^i \neq 0$.

Taking Ɓ-covariant derivative for (3.7) with respect to x^m, in the since of Berwald and using (1.9) we get

(3.8) $$\mathcal{B}_m(\mathcal{B}_l U_{jkh}^i) = (\mathcal{B}_m\lambda_l)U_{jkh}^i + \lambda_l\,\mathcal{B}_m U_{jkh}^i) + (\mathcal{B}_m\mu_l)\,(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk})$$
$$-2\,\mu_l\, y^r\mathcal{B}_r(\delta_k^i\, C_{jhm} - \delta_h^i\, C_{jkm}) + \frac{1}{4}(\mathcal{B}_m\eta_l)\,(H_k^i\, g_{jh} - H_h^i\, g_{jk})$$
$$+\frac{1}{4}\eta_l[(\mathcal{B}_m H_k^i)g_{jh} - (\mathcal{B}_m H_h^i)g_{jk}] - \frac{1}{2}\eta_l\, y^r\mathcal{B}_r\,[H_k^i\,(\mathcal{B}_r C_{jhm}) - H_h^i\,(\mathcal{B}_r C_{jkm}\,)]\,.$$

Using (3.7) in (3.8), we get

(3.9) $$\mathcal{B}_m\mathcal{B}_l U_{jkh}^i = w_{lm}\, U_{jkh}^i + v_{lm}(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk}) + \frac{1}{4}c_{lm}(H_k^i\, g_{jh} - H_h^i\, g_{jk})$$
$$- 2\,\mu_l\, y^r\mathcal{B}_r(\delta_k^i\, C_{jhm} - \delta_h^i\, C_{jkm}) + \frac{1}{4}\,\eta_l[(\mathcal{B}_m H_k^i)g_{jh} - (\mathcal{B}_m H_h^i)g_{jk}]$$
$$-\frac{1}{2}\eta_l\, y^r(H_k^i\,\mathcal{B}_r\, C_{jhm} - H_h^i\,\mathcal{B}_r C_{jkm})\ .$$

Where $w_{lm} = \mathcal{B}_m\lambda_l + \lambda_l\,\lambda_m$, $v_{lm} = \lambda_l\,\mu_m + \mathcal{B}_m\mu_l$ and $c_{lm} = \lambda_l\eta_l + \mathcal{B}_m\eta_l$ are non-zero covariant tensors fields of second order and μ_l and η_l are non-zero covariant vector fields of first order.

Thus, we conclude

Theorem 3.2. A $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$,may characterized by the condition (3.9).

Suppose that $H^i_k\, g_{jh} = H^i_{jkh}$ in (3.9), we get

(3.10) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = w_{lm}U^i_{jkh} + v_{lm}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) + \tfrac{1}{4}c_{lm}(H^i_{jkh} - H^i_{jhk})$$
$$-2\,\mu_l\, y^r\mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h\, C_{jkm}) + \tfrac{1}{4}\eta_l[(\mathcal{B}_m\, H^i_k\,)g_{jh} - (\mathcal{B}_m H^i_h\,)g_{jk}]$$
$$-\tfrac{1}{2}\eta_l\, y^r(H^i_k\,\mathcal{B}_r\, C_{jhm} - H^i_h\,\mathcal{B}_r C_{jkm}) .$$

By using (1.21) in (3.9), we get

(3.11) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = w_{lm}\, U^i_{jkh} + \tfrac{1}{2}c_{lm}\, H^i_{jkh} + v_{lm}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$$
$$-2\,\mu_l\, y^r\,\mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h\, C_{jkm}) + \tfrac{1}{4}\eta_l[(\,\mathcal{B}_m H^i_k\,)g_{jh} - (\mathcal{B}_m H^i_h\,)g_{jk}]$$
$$-\tfrac{1}{2}\eta_l\, y^r(H^i_k\,\mathcal{B}_r C_{jhm} - H^i_h\,\mathcal{B}_r C_{jkm}) .$$

Thus, we conclude

Corollary 3.3. In $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$, the relationship between the h(v)-curvature tensor U^i_{jkh} and Berwald's curvature tensor H^i_{jkh} may given by (3.11).

Let's look at a Finsler space F_n where the h(v) curvature tensor U^i_{jkh} meet the requirements listed below:

(3.12) $$\mathcal{B}_l U^i_{jkh} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) + \tfrac{1}{4}\eta_l(H^i_k g_{jh} - H^i_h g_{jk}) \;,\; U^i_{jkh} \neq 0 .$$

Taking Ɓ-covariant derivative for (3.12) with respect to x^m, in the since of Berwald and using (1.9) we get

(3.13) $$\mathcal{B}_m\,(\mathcal{B}_l U^i_{jkh}) = (\mathcal{B}_m\lambda_l)U^i_{jkh} + \lambda_l(\mathcal{B}_m U^i_{jkh}) + (\mathcal{B}_m\mu_l)\,(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$$
$$-\,2\,\mu_l\, y^r\mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h\, C_{jkm}) + \tfrac{1}{4}(\mathcal{B}_m\eta_l)(R^i_k\, g_{jh} - R^i_h\, g_{jk})$$
$$+\tfrac{1}{4}\,\eta_l[(\mathcal{B}_m\, R^i_k\,)g_{jh} - (\mathcal{B}_m R^i_h\,)g_{jk}] - \tfrac{1}{2}\eta_l\, y^r\mathcal{B}_r\,(R^i_k\,\mathcal{B}_r C_{jhm} - R^i_h\,\mathcal{B}_r C_{jkm}) .$$

By using (3.12) in (3.13), we get

(3.14) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = w_{lm}\, U^i_{jkh} + v_{lm}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) + \tfrac{1}{4}c_{lm}\,(R^i_k\, g_{jh} - R^i_h\, g_{jk})$$
$$-\,2\,\mu_l\, y^r\,\mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h\, C_{jkm}) + \tfrac{1}{4}\eta_l[(\mathcal{B}_m\, R^i_k\,)g_{jh} - (\mathcal{B}_m R^i_h\,)g_{jk}]$$
$$-\,\tfrac{1}{2}\eta_l\, y^r(R^i_k\,\mathcal{B}_r C_{jhm} - R^i_h\,\mathcal{B}_r C_{jkm}) ,$$

Where $w_{lm} = \mathcal{B}_m\lambda_l + \lambda_l\,\lambda_m$, $v_{lm} = \lambda_l\,\mu_m + \mathcal{B}_m\mu_l$ and $c_{lm} = \lambda_l\eta_l + \mathcal{B}_m\eta_l$ are non-zero covariant tensors fields of second order.

Thus, we may conclude

Theorem 3.3. A $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$,may characterized by the condition (3.14).

Let's look at a Finsler space F_n where h(v)-curvature tensor U^i_{jkh} meet the requirements listed below:

(3.15) $$\mathcal{B}_l U^i_{jkh} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) + \tfrac{1}{4}\eta_l(N^i_k g_{jh} - N^i_h g_{jk}) \;,\; U^i_{jkh} \neq 0 .$$

Taking Ɓ-covariant derivative for (3.15) with respect to x^m, in the since of Berwald and using (1.9) we get

(4.16) $$\mathcal{B}_m(\mathcal{B}_l U^i_{jkh}) = (\,\mathcal{B}_m\lambda_l\,)U^i_{jkh} + \lambda_l(\mathcal{B}_m U^i_{jkh}) + (\,\mathcal{B}_m\mu_l)\,(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk})$$
$$-\,2\,\mu_l\, y^r\mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h\, C_{jkm}) + \tfrac{1}{4}(\,\mathcal{B}_m\eta_l)\,(\,N^i_k\, g_{jh} - N^i_h\, g_{jk}\,)$$
$$+\tfrac{1}{4}\eta_l\,[(\mathcal{B}_m N^i_k\,)g_{jh} - (\mathcal{B}_m N^i_h\,)g_{jk}] - \tfrac{1}{2}\eta_l\, y^r\mathcal{B}_r(N^i_k\,\mathcal{B}_r C_{jhm} - N^i_h\,\mathcal{B}_r C_{jkm}) .$$

Using (3.15) in (3.16), we get

(3.17) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = w_{lm}\, U^i_{jkh} + v_{lm}(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}) + \tfrac{1}{4}c_{lm}(N^i_k\, g_{jh} - N^i_h\, g_{jk})$$
$$-2\,\mu_l\, y^r\,\mathcal{B}_r(\delta^i_k\, C_{jhm} - \delta^i_h\, C_{jkm}) + \tfrac{1}{4}\,\eta_l\,[(\mathcal{B}_m\, N^i_k\,)g_{jh} - (\,\mathcal{B}_m N^i_h\,)g_{jk}]$$
$$-\tfrac{1}{2}\eta_l\, y^r(N^i_k\,\mathcal{B}_r C_{jhm} - N^i_h\,\mathcal{B}_r C_{jkm}) .$$

Where $w_{lm} = \mathcal{B}_m\lambda_l + \lambda_l\,\lambda_m$, $v_{lm} = \lambda_l\,\mu_m + \mathcal{B}_m\mu_l$ and $c_{lm} = \lambda_l\eta_l + \mathcal{B}_m\eta_l$ are non-zero covariant tensors of second order, respectively.

Theorem 3.3. A $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$,may characterized by the condition (3.17).

Suppose that, $N_k^i\, g_{jh} = N_{jkh}^i$ in (3.16), we get

(3.18) $$\mathcal{B}_m\mathcal{B}_l U_{jkh}^i = w_{lm}\, U_{jkh}^i + v_{lm}\big(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk}\big) + \tfrac{1}{4} c_{lm}\big(N_{jkh}^i - N_{jhk}^i\big) -2\,\mu_l\, y^r\mathcal{B}_r\big(\delta_k^i\, C_{jhm} - \delta_h^i\, C_{jkm}\big) + \tfrac{1}{4}\,\eta_l\big[(\mathcal{B}_m N_k^i)g_{jh} - (\mathcal{B}_m N_h^i)g_{jk}\big] -\tfrac{1}{2}\eta_l\, y^r\big(N_k^i\,\mathcal{B}_r C_{jhm} - N_h^i\,\mathcal{B}_r C_{jkm}\big) .$$

By using (1.31) in (3.18), we get

(3.19) $$\mathcal{B}_m\mathcal{B}_l U_{jkh}^i = w_{lm} U_{jkh}^i + \tfrac{1}{2} c_{lm}\, N_{jkh}^i + v_{lm}\big(\delta_k^i\, g_{jh} - \delta_h^i\, g_{jk}\big) - 2\,\mu_l\, y^r\,\mathcal{B}_r\big(\delta_k^i\, C_{jhm} - \delta_h^i\, C_{jkm}\big) + \tfrac{1}{4}\eta_l\big[(\mathcal{B}_m\, N_k^i)g_{jh} - (\mathcal{B}_m N_h^i)g_{jk}\big] - \tfrac{1}{2}\eta_l\, y^r\big(N_k^i\,\mathcal{B}_r C_{jhm} - N_h^i\,\mathcal{B}_r C_{jkm}\big) .$$

Thus, we conclude

Theorem 3.5. A $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$,the relationship between h(v)-curvature tensor U_{jkh}^i and the normal projective curvature tensor N_{jkh}^i is given by (3.19).

Contracting the indices i and h in (3.19), using (1.15b), (1.32), (1.4c) and (1.4g), we get

(3.20) $$\mathcal{B}_m\mathcal{B}_l U_{jk} = w_{lm} U_{jk} + \tfrac{1}{2} c_{lm}\, N_{jk} + (1-n)\big(v_{lm} g_{jk} - 2\,\mu_l\, y^r\,\mathcal{B}_r C_{jkm}\big) +\tfrac{1}{4}\,\eta_l\big[(\mathcal{B}_m\, N_k^s)g_{js} - (\mathcal{B}_m N_s^s)g_{jk}\big] - \tfrac{1}{2}\eta_l\, y^r\big(N_k^s\,\mathcal{B}_r C_{jsm} - N_s^s\mathcal{B}_r C_{jkm}\big) .$$

Thus, we conclude

Corollary 3.4. The equation (3.20) holds in $G^{2nd}\mathcal{B}_h U\text{-}BRF_n$.

Conclusions

The introduction of generalized U-birecurrent Finsler spaces has significantly enriched the field of Finsler geometry. Our research has revealed several novel properties of these spaces and has expanded our understanding of their geometric structure. The results presented in this paper open up new avenues for further research.

Recommendations for Future Research

- Explore the physical applications of generalized U-birecurrent Finsler spaces.
- Investigate the relationship between generalized U-birecurrent Finsler spaces and other geometric structures, such as conformal Finsler geometry.
- Develop numerical methods for studying the properties of generalized U–birecurrent Finsler spaces.

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فئة جديدة من فضاءات فنسلر المعممة ذات التكرار الثنائي-U وخصائص انحنائها

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الملخص: في هذا البحث، نقدم فئة جديدة من فضاءات فنسلر المعممة ذات التكرار الثنائي-U ، والمُعرَّفة بدلالة مشتقات بيروالد التوافقية من الرتبة الثانية للتنسور الانحنائي $h(v)$. يوسّع هذا التركيب الجديد المفاهيمَ القائمة لفضاءات فنسلر التكرارية وذات التكرار الثنائي، وذلك من خلال إدخال شروط تفاضلية من رتب أعلى. وقد تم إثبات عدة خصائص أساسية للفضاءات المقترحة، بما في ذلك العلاقات بين التنسور الانحنائي $h(v)$ وغيره من تنسورات الانحناء المهمة، مثل تنسورات انحناء كارتان، وتنسور انحناء بيروالد، وتنسور الانحناء الإسقاطي العادي. علاوة على ذلك، تم اشتقاق مجموعة من النظريات والنتائج التابعة لتوصيف السلوك الهندسي لهذه الفضاءات تحت شروط التكرار المعمم. وتوفر النتائج المتحصَّل عليها إطارًا أوسع لدراسة البنى الانحنائية في هندسة فنسلر ، كما تسهم في التطوير المستمر للتحليل الهندسي من الرتب العليا.

كلمات مفتاحيه: هندسة فنسلر؛ الفضاء المعمم ذو التكرار الثنائي؛ التكرار الثنائي-U؛ مشتقة بيروالد التوافقية؛ التنسور الانحنائي؛ تنسور انحناء كارتان؛ تنسور الانحناء الإسقاطي.