On Generalized Nth -Order Recurrent Finsler Spaces Associated with Cartan's Curvature Tensor

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Abstract: This paper investigates the geometric structure of a Finsler space F_n admitting a generalized n^{th} -order recurrence property relative to Cartan's third curvature tensor field R^i_{jkh}. We define a new class of Finsler spaces, denoted as $GR^h\text{-}n^{th}RF_n$, where the h-covariant derivative of the nth order of the curvature tensor R^i_{jkh} satisfies a specific recurrence condition involving multiple recurrence tensor fields. Throughout this study, we derive several fundamental geometric identities, including Bianchi and Veblen-type identities, within the framework of this generalized space. Furthermore, we establish necessary and sufficient conditions under which Cartan's fourth curvature tensor K^i_{jkh} and the Ricci tensor exhibit recurrence properties. The results demonstrate that the curvature scalar and the deviation tensor associated with Cartan's third curvature tensor are non-vanishing in $GR^h\text{-}n^{th}RF_n$. These findings extend the existing theory of recurrent Finsler spaces to higher-order differential manifolds.

1. Introduction

The theory of recurrent spaces has been a focal point in Finsler geometry, particularly following the foundational works of Berwald and Cartan. While many researchers have explored recurrent and birecurrent Finsler spaces, most existing literature remains confined to second or third-order recurrence properties. However, a comprehensive framework for generalized n^{th}-order recurrence involving Cartan's third curvature tensor field R^i_{jkh} has not been fully established.

This paper bridges this gap by introducing the concept of Generalized $R^h\text{-}n^{th}RF_n$ Recurrent Finsler Spaces $GR^h\text{-}n^{th}RF_n$. Unlike standard recurrence, our approach utilizes a generalized condition where the h-covariant derivative of the nth order is expressed through a combination of multiple recurrence tensors and metric components. By investigating these higher-order structures, we provide a more general geometric perspective that encompasses previous lower-order models as special cases.

Literature Review:

The geometric framework of Finsler spaces and their foundational differential properties were extensively established by Rund [2], providing the basis for subsequent developments in curvature tensor analysis. Over the past decade, the study of recurrent and generalized recurrent Finsler structures has gained significant momentum. Al-Qashbari and Qasem [6] initiated investigations into generalized BR-trirecurrent spaces, while further properties of Weyl's projective and generalized curvature tensors of the second order were explored by Al-Qashbari [7, 8]. Recent advancements have shifted toward higher-order recurrence; for instance, Al-Qashbari et al. [11, 17] studied fourth-order and Kh-generalized recurrent spaces, extending the theory to more complex manifolds. In 2024, significant contributions were made regarding R-projective curvature tensors [12] and the interaction of concircular curvature tensors under Lie derivatives [14, 15]. Furthermore, the decomposition of curvature tensors using Berwald's and Cartan's higher-order derivatives has been rigorously analyzed [4, 5], highlighting the importance of mixed covariant derivatives in the Cartan sense [13]. These studies collectively emphasize that while lower-order recurrence is well-documented, the exploration of generalized n^{th}-order recurrence associated specifically with Cartan's third curvature tensor remains an open and fertile area of research.

Cartan in his second postulate, represented the variation of an arbitrary vector field X^i under the infinitesimal change of its line element (x, y) to $(x + dx, y + dy)$ by means of covariant (absolute) differential given by

(1.1) $DX^i = dX^i + X^j \left(C^i_{jk} \, dy^k + \Gamma^i_{jk} \, dx^k \right)$, where

(1.2)   a) $\Gamma^i_{jk} = \gamma^i_{jk} - C^i_{mk} G^m_j + g^{ih} C_{jkm} G^m_h$,

          b) $G^i = \frac{1}{2} \gamma^i_{jk} y^j y^k$ and   c) $G^i_j = \dot{\partial}_j G^i$.

The function G^i is positively homogeneous of degree two in the directional argument.

Eliminating dy^k from (1.1) and in terms of the absolute differential of l^i, Cartan deduced

(1.3) $DX^i = F X^i|_k D l^k + X^i_{|k} dx^k + y^k \left(\dot{\partial}_k X^i \right) \frac{dF}{F}$, where

(1.4)   a) $X^i|_k = \dot{\partial}_k X^i + X^r C^i_{rk}$, b) $X^i_{|k} = \partial_k X^i + X^r \Gamma^{*i}_{rk} - \left(\dot{\partial}_m X^i \right) \Gamma^m_{sk} y^s$,

and   c) $\Gamma^{*i}_{rk} = \Gamma^i_{rk} - C^i_{mr} \Gamma^m_{sk} y^s$.

The function Γ^{*i}_{rk} defined by (1.4c) is connection parameter of Cartan, this is symmetric in the lower indices r and k and positively homogeneous of degree zero in the directional argument and satisfies:

(1.5) $g_{ih} \Gamma^{*i}_{rk} = \Gamma^*_{rhk}$.

The equations (1.4a) and (1.4b) give two processes of covariant differentiation called v-covariant differentiation (Cartan's first kind covariant differentiation) and h-covariant differentiation

(Cartan's second kind covariant differentiation), respectively. So $X^i|_k$ and $X^i_{|k}$ are respectively v-covariant derivative and h-covariant derivative of the vector field X^i. We note that this notation for covariant differentiations was used by Cartan and followed by Rund and Matsumoto calls these derivatives as " v-covariant derivative " and " h-covariant derivative ", respectively and his symbols for covariant differentiations are similar to that of Cartan with only difference that $\frac{1}{F} X^i|_k$ of Cartan coincides with $X^i|_k$ of Matsumoto due to this change we have an extra F in first term of the right hand side of the equation (1.5). K. Yano denoted $\frac{1}{F} X^i|_k$ and $X^i|_k$ by $\dot{\nabla}_k X^i$ and $\nabla_j X^i$, respectively.

The metric tensor g_{ij} and the associate metric tensor g^{ij} are related by

(1.6) $\qquad g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$,

The quantities g_{ij}, g^{ij} and δ_j^i are satisfies

(1.7) $\qquad$ a) $\quad g_{ij}\, g^{ij} = n \quad$ and $\quad$ b) $\quad \delta_j^i\, g_{ik} = g_{jk}$.

The vector y_i satisfies relation

(1.8) $\qquad y_i\, y^i = F^2$

The vectors y_i and δ_k^i also satisfy the following relations

(1.9) $\qquad$ a) $\quad \delta_k^i\, y^k = y^i \quad$, $\qquad$ b) $\quad \delta_j^i\, g^{jk} = g^{ik}$ and $\quad$ c) $\quad g_{ij}\, y^j = y_i$.

By using Euler's theorem, the C_{ijk} and C_{jk}^i tensors satisfy, the following identities

(1.10) $\qquad$ a) $\quad C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0 \quad$ and $\qquad$ b) $\quad C_{jk}^i\, y^j = C_{kj}^i\, y^j = 0$.

The metric tensor g_{ij} and the associate metric tensor g^{ij} are covariant constant with respect to both processes

(1.11) $\qquad$ a) $\quad g_{ij|m} = 0 \quad$ and $\qquad$ b) $\quad g^{ij}_{|m} = 0$.

The vectors y^i, y_i are vanish under h-covariant differentiation

(1.12) $\qquad$ a) $\quad y_{i|m} = 0 \quad$ and $\qquad$ b) $\quad y^i_{|m} = 0$.

The h-curvature tensor R^i_{jkh} (Cartan's third curvature tensor), is defined by

(1.13) $\qquad R^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + \left(\dot{\partial}_l \Gamma^{*i}_{jk} \right) G^l_h + C^i_{jm}\left(\dot{\partial}_k G^m_h - G^m_{kl}\, G^l_h \right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh}$

$\qquad\qquad - [\dot{\partial}_k \Gamma^{*i}_{jh} + \left(\dot{\partial}_l \Gamma^{*i}_{jh} \right) G^l_k + C^i_{jm}\left(\dot{\partial}_h G^m_k - G^m_{hl}\, G^l_k \right) + \Gamma^{*i}_{mh}\, \Gamma^{*m}_{jk}]$.

The h-curvature tensor R^i_{jkh} is positively homogeneous of degree -1 in the directional argument and skew-symmetric in the last two lower indices h and k, i.e.

(1.14) $\qquad R^i_{jkh} = - R^i_{jhk}$

and this tensor satisfies the following relation too

(1.15) $\qquad R^i_{jkh} = K^i_{jkh} + C^i_{js}\, K^s_{rkh}\, y^r$.

The associate curvature tensor R_{ijkh} of the curvature tensor R^i_{jkh} is given by

(1.16)   a) $R_{ijkh} = g_{rj} R^r_{ikh}$ and   b) $R_{jrkh} \, g^{ir} = R^i_{jkh}$.

The R-Ricci tensor R_{jk} , the curvature scalar R and the deviation tensor R^i_j related by

(1.17)   a) $R^i_{jki} = R_{jk}$,   b) $R_{jk} y^k = R_j$,   c) $R_{jk} y^j = H_k$,

d) $R_{jk} \, g^{jk} = R$ and   e) $R^i_i = R$.

The curvature tensor R^i_{jkh} and the associate tensor R^r_h are satisfies the relations

(1.18)   a) $R^i_{jkh} y^j = K^i_{jkh} y^j = H^i_{kh}$

and   b) $R^r_h = R^r_{ikh} \, g^{ik}$.

Cartan's connection parameter Γ^{*i}_{jk} and Berwald's connection parameter G^i_{jm} given by

(1.19)   a) $\dot{\partial}_k G^i_h = G^i_{kh}$ and   b) $G^i_k = \Gamma^{*i}_{sk} \, y^s$.

(1.20) $\left(\dot{\partial}_h \Gamma^{*i}_{jk} \right) y^h = \Gamma^{*i}_{jkh} \, y^h = G^i_{jkh} \, y^h = 0$.

The tensor P^i_{kh} is called v(hv)-torsion tensor and its associate tensor P_{kjh} is given by

(1.21)   a) $\Gamma^{*i}_{jkh} y^j = P^i_{kh}$,   b) $y_i \Gamma^{*i}_{kjh} = -P_{kjh}$ and   c) $g_{rj} P^r_{kh} = P_{kjh}$.

the tensors H^i_{jkh} and H^i_{kh} form the components of tensors and defined by

(1.22) $H^i_{jkh} = \dot{\partial}_h G^i_{jk} + G^r_{jk} G^i_{rh} + G^i_{rjh} G^r_k - \dot{\partial}_k G^i_{jh} - G^r_{jh} G^i_{rk} - G^i_{rjk} G^r_h$, and

(1.23) $H^i_{kh} = \dot{\partial}_h G^i_k + G^r_k C^i_{rh} - \dot{\partial}_k G^i_h - G^r_h C^i_{rk}$.

The formula (1.23) called generalized Ricci identity or Ricci commutation formula.

(1.24)   a) $H_k = H^i_{ki}$ and   b) $H_k y^k = (n-1)H$, where

H^i_{hk} and H^i_k are called H-Ricci tensor and the curvature scalar, respectively and defined by

(1.24) $H^i_{hk} y^h = H^i_k$.

(1.25) $K^i_{rkj} = \partial_j \Gamma^{*i}_{kr} + \left(\dot{\partial}_l \Gamma^{*i}_{rj} \right) G^l_k + \Gamma^{*i}_{mj} \Gamma^{*m}_{kr} - \partial_k \Gamma^{*i}_{jr} - \left(\dot{\partial}_l \Gamma^{*i}_{rk} \right) G^l_j - \Gamma^{*i}_{mk} \Gamma^{*m}_{jr}$.

The tensor K^i_{rkj} as defined (1.25) above is called Cartan's fourth curvature tensor, this tensor is positively homogeneous of degree zero.

The curvature tensor K^i_{jkh} satisfies the following relation too

(1.26)   a) $g_{rj} K^r_{ikh} = K_{ijkh}$ and   b) $K^i_{jkh} y^j = H^i_{kh}$.

he associates curvature tensor K_{ijkh} satisfies the condition

(1.27) $K_{jikh} + K_{ijkh} = -2 \, C_{ijs} K^s_{rkh} y^r$.

Ricci tensor K_{jk} and the curvature vector K_j of the curvature tensor K^i_{jkh} are given by

(1.28)   a) $K^i_{jki} = K_{jk}$

and   b) $K_{jk} y^k = K_j$.

2. On Generalized R^h-Recurrent Finsler Space of N$^{\text{th}}$ order

In this section, we analyze the structural properties of the defined Generalized R^h-n^{th} Recurrent Finsler Space GR^h-$n^{th}RF_n$. The core of our derivation starts from the generalized recurrence condition for Cartan's third curvature tensor R^i_{jkh} .

Definition 2.1: The GR^h-$n^{th}RF_n$ Space

A Finsler space F_n is defined as a Generalized R^h-n^{th} order recurrent space (denoted briefly as GR^h-$n^{th}RF_n$) if its Cartan's third curvature tensor R^i_{jkh} is non-null and satisfies the following condition:

$$(2.1) \quad R^i_{jkh|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} R^i_{jkh} + \mu_{m_1 m_2 ... m_n} \left(\delta^i_h \, g_{jk} - \delta^i_k \, g_{jh} \right)$$

$$+ \frac{1}{4} \delta_{m_1 m_2 ... m_n} \left(R^i_k \, g_{jh} - R^i_h \, g_{jk} \right) \ ,$$

where $|m_1|m_2|...|m_n$ denotes the h-covariant differentiation of the nth order with respect to x^m, and λ, μ and δ are the recurrence tensor fields.

Since the metric tensor is a covariant constant, then transvecting of the condition (2.1) by g_{ip} , using (1.11a), (1.16a) and (1.7b), we get

$$(2.2) \quad R_{jpkh|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} R_{jpkh} + \mu_{m_1 m_2 ... m_n} \left(g_{hp} \, g_{jk} - g_{kp} \, g_{jh} \right)$$

$$+ \frac{1}{4} \delta_{m_1 m_2 ... m_n} \left(R_{kp} \, g_{jh} - R_{hp} \, g_{jk} \right) R_{kp} \quad , \text{where} \quad g_{ip} R^i_k = R_{kp} \ .$$

Conversely, the transvection of the condition (2.2) by g^{ip} , yields the condition (2.1). Thus, the condition (2.2) is equivalent to the condition (2.1). Therefore, a generalized R^h-n^{th} order space may be characterized by the condition (2.2).

Therefore, we conclude

Theorem 2.1: Characterization via Associate Curvature Tensor

A generalized R^h-n^{th} order space GR^h-$n^{th}RF_n$ can be characterized by the following equivalent condition involving the associate curvature tensor R_{jpkh} .

Let us consider an GR^h-$n^{th}RF_n$, which characterized by the condition (2.1).

Contracting the indices i and h in (2.1), using (1.17a), (1.6), (1.17e) and (1.7b), we get

$$(2.3) \quad R_{jk|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} R_{jk} + (n-1) \, \mu_{m_1 m_2 ... m_n} g_{jk}$$

$$+ \frac{1}{4} \delta_{m_1 m_2 ... m_n} \left(R^i_k \, g_{ji} - R g_{jk} \right) \quad .$$

Transvecting the equation (2.3) by y^k , using (1.12b), (1.17b) and (1.9c), we get

$$(2.4) \quad R_{j|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} R_j + (n-1) \mu_{m_1 m_2 ... m_n} y_j$$

$$+ \frac{1}{4} \delta_{m_1 m_2 ... m_n} \left(R^i_k \, g_{ji} y^k - R y_j \right) \quad .$$

Further, transvecting (2.1) by g^{jk} , using (1.11b), (1.18b) and in view of (1.6), we get

$$(2.5) \quad R^i_{h|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} R^i_h + (n-1) \, \mu_{m_1 m_2 ... m_n} \delta^i_h$$

$$+\frac{1}{4}\delta_{m_1 m_2 \dots m_n}\left(R_k^i g_{jh} - R_h^i \delta_h^i\right) \ .$$

Contracting the indices i and h in the condition (2.5), (1.17e) and using (1.6), we get

$$(2.6) \qquad R_{|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} R + (n^2 - n)\,\mu_{m_1 m_2 \dots m_n} + \frac{1}{4}\delta_{m_1 m_2 \dots m_n}\left(R_k^i g_{ji} - nR_h^i\right) \ .$$

Also, by transvecting the equation (2.3) by g^{jk} , using (1.11b), (1.17d) and in view of (1.6), we get the condition (2.6).

The conditions (2.3), (2.4), (2.5) and (2.6), show that, Ricci tensor R_{jk} , the curvature vector R_j , the deviation tensor R_h^i and the curvature scalar R (all for Cartan's third curvature tensor R_{jkh}^i) of a generalized R^h-nth order space cannot vanish, because the vanishing of them imply the vanishing of the covariant tensors fields $\mu_{m_1 m_2 \dots m_n}$, i.e. $\mu_{m_1 m_2 \dots m_n} = 0$,   a contradiction. Thus, we conclude

Theorem 2.2: Non-vanishing of Fundamental Tensors

In a $GR^h\text{-}n^{th}RF_n$ space, the following geometric entities associated with Cartan's third curvature tensor cannot vanish:

- The **Ricci tensor** R_{jk} .
- The **curvature vector** R_j .
- The **deviation tensor** R_h^i .
- The **curvature scalar** R .

Note: The vanishing of these tensors would imply the vanishing of the recurrence tensor field $\mu_{m_1 m_2 \dots m_n}$, which contradicts the space definition.

Transvecting the condition (2.3) by y^j , using (1.12b), (1.17c) and (1.9c), we get

$$(2.7) \qquad H_{k|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n}\, H_k + (n-1)\,\mu_{m_1 m_2 \dots m_n}y_k$$

$$+\frac{1}{4}\delta_{m_1 m_2 \dots m_n}\left(R_k^i\, y_i - Ry_k\right) \ .$$

The condition (2.7), shows that, the curvature vector H_k of a generalized R^h- nth order space cannot vanish, because the vanishing of it would imply the vanishing of the covariant tensor's fields $\mu_{m_1 m_2 \dots m_n}$, i.e. $\mu_{m_1 m_2 \dots m_n} = 0$, a contradiction.

Transvecting the condition (2.7) by y^k, using (1.12b), (1.24b) and (1.8), we get

$$(2.8) \qquad H_{|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n}\, H + \mu_{m_1 m_2 \dots m_n}\, F^2 + \frac{1}{4}\delta_{m_1 m_2 \dots m_n}\left(R_k^i\, y_i y^k - RF^2\right) \ .$$

Thus, we conclude

Theorem 2.3: Properties of the Curvature Vector and Scalar

In a $GR^h\text{-}n^{th}RF_n$, both the curvature vector H_k and the curvature scalar H are non-vanishing. This is due to the fact that their vanishing would necessitate $\mu_{m_1 m_2 \dots m_n} = 0$, leading to a logical contradiction.

Now, we have seen, in a generalized R^h-nth order space, Ricci tensor R_{jk} of Cartan's third curvature tensor R_{jkh}^i satisfies the condition (2.3). Conversely, if Ricci tensor R_{jk} of a Finsler space satisfy the condition (2.3), then it need not be the space is a generalized R^h-nth order space. However, the converse is true if the dimension of a Finsler space is 3 or the space is $R3$-like. The proof of this fact as follows:

We know that the associate curvature tensor R_{ijkh} (of Cartan's third curvature tensor R_{jkh}^i) for three dimensioned Finsler space is of the form

(2.9) $\qquad R_{ijkh} = g_{ik} L_{jh} + g_{jh} L_{ik} - g_{ih} L_{jk} - g_{jk} L_{ih}$, where

(2.10) $\qquad L_{ik} = \frac{1}{(n-2)}\left(R_{ik} - \frac{r}{2} g_{ik}\right)$ and

(2.11) $\qquad r = \frac{1}{(n-1)} R_j^i$.

Transvecting the condition (2.3) by g^{jp} , using (1.11b), (1.18b) and (1.6), we get

(2.12) $\qquad R_{k|m_1|m_2|...|m_n}^p = \lambda_{m_1 m_2 ... m_n} R_k^p + (n-1)\mu_{m_1 m_n ... m_n} \delta_k^p$

$\qquad\qquad + \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R_k^i \delta_i^p - R \delta_k^p\right)$.

Contracting the indices p and k in the condition (2.12), using (2.11) and (1.6), we get

(2.13) $\qquad r_{|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} r + n \mu_{m_1 m_2 ... m_n} + \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R_k^i \delta_i^k - n R\right)$.

Taking the h-covariant derivative of the condition (2.10) with respect to x^m to nth order and using (1.6), we get

(2.14) $\qquad L_{ik|m_1|m_2|...|m_n} = \frac{1}{(n-2)}\left(R_{ik|m_1|m_2|...|m_n} - \frac{1}{2} r_{|m_1|m_2|...|m_n} g_{ik}\right)$.

Using the conditions (2.3) and (2.13) in (2.14), we get

$$L_{ik|m_1|m_2|...|m_n} = \frac{1}{(n-2)}\left(\lambda_{m_1 m_2 ... m_n} R_{jk} + (n-1)\mu_{m_1 m_2 ... m_n} g_{jk} + \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R_k^i g_{ji} - R g_{jk}\right)\right)$$

$$- \frac{1}{2(n-2)}\left(\lambda_{m_1 m_2 ... m_n} r + n \mu_{m_1 m_2 ... m_n} + \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R_k^i g_{ji} - R g_{jk}\right)\right) g_{ik} ,$$

Or we can be written as

$$L_{ik|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n}\left[\frac{1}{(n-2)}\left(R_{ik} - \frac{1}{2} r g_{ik}\right)\right] + \frac{1}{2}\mu_{m_1 m_2 ... m_n} g_{ik}$$

$$+ \frac{1}{4(n-2)}\delta_{m_1 m_2 ... m_n}\left(R_k^i g_{ji} - R g_{jk}\right) .$$

In view of the condition (2.10), the above equation implies to

(2.15) $\qquad L_{ik|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} L_{ik} + \frac{1}{2}\mu_{m_1 m_2 ... m_n} g_{ik}$

$\qquad\qquad + \frac{1}{4(n-2)}\delta_{m_1 m_2 ... m_n}\left(R_k^i g_{ji} - R g_{jk}\right)$.

This gives the h-covariant derivative of Ricci tensor L_{ik} in generalized R^h-nth order space.

The h-covariant derivative of the condition (2.9) with respect to x^m to nth order and using (1.11a), we get

$$R_{ijkh|m_1|m_2|\ldots|m_n} = g_{ik}\, L_{jh|m_1|m_2|\ldots|m_n} + g_{jh}\, L_{ik|m_1|m_2|\ldots|m_n}$$
$$-g_{ih}\, L_{jk|m_1|m_2|\ldots|m_n} - g_{jk}\, L_{ih|m_1|m_2|\ldots|m_n} \quad .$$

In view of the condition (2.15), the above equation implies

$$R_{ijkh|m_1|m_2|\ldots|m_n} = \lambda_{m_1 m_2 \ldots m_n}\big(g_{ik} L_{jh} + g_{jh} L_{ik} - g_{ih} L_{jk} - g_{jk} L_{ih}\big)$$
$$+\mu_{m_1 m_2 \ldots m_n}\big(g_{ik} g_{jh} - g_{ih} g_{jk}\big) + \tfrac{1}{4(n-2)}\delta_{m_1 m_2 \ldots m_n}\big(g_{ik} g_{jh} - g_{ih} g_{jk}\big)\big(R^i_k g_{ji} - R g_{jk}\big) .$$

In view of the condition (2.9), the above equation implies

$$(2.16) \qquad R_{ijkh|m_1|m_2|\ldots|m_n} = \lambda_{m_1 m_2 \ldots m_n} R_{ijkh} + \mu_{m_1 m_2 \ldots m_n}\big(g_{ik} g_{jh} - g_{ih} g_{jk}\big)$$
$$+ \tfrac{1}{4(n-2)}\delta_{m_1 m_2 \ldots m_n}\big(g_{ik} g_{jh} - g_{ih} g_{jk}\big)\big(R^i_k g_{ji} - R g_{jk}\big) \quad .$$

This shows that, the associate curvature tensor R_{ijkh} (of Cartan's third curvature tensor R^i_{jkh}) is a generalized h-recurrent.

In view of (1.16a), the above condition implies (2.1). That is, the h-covariant derivative of the condition (2.9) with respect to x^m to nth order and in view of (1.8a), gives (2.1). This shows that, a three-dimensional Ricci generalized R^h-nth order space is necessarily generalized R^h-recurrent space.

Matsumote introduced a Finsler space $F_n (n > 3)$ for which the tensor R_{ijkh} satisfying (2.9), and call it $R3$-like Finsler space F_n .

This, leads to

Theorem 2.4: Sufficiency in Three-Dimensional and R3-like Spaces

A GR^h-$n^{th}RF_n$ is always a Ricci generalized R^h-recurrent space. While the converse is not generally true, it holds if the Finsler space is three-dimensional or an R3-like space.

Now, taking the h-covariant derivative for (1.13), with respect to x^m to nth order, we get

$$(2.17) \qquad R^i_{jkh|m_1|m_2|\ldots|m_n} = \big[\, \dot\partial_h \Gamma^{*i}_{jk} + \big(\dot\partial_s \Gamma^{*i}_{jk}\big)G^s_h + C^i_{jm}\big(\dot\partial_k G^m_h - G^m_{kt} G^t_h\big) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh}$$
$$- \dot\partial_k \Gamma^{*i}_{jh} - \big(\dot\partial_s \Gamma^{*i}_{jh}\big)G^s_k - C^i_{jm}\big(\dot\partial_h G^m_k - G^m_{ht} G^t_k\big) - \Gamma^{*i}_{mh} \Gamma^{*m}_{jk}\big]_{|m_1|m_2|\ldots|m_n} .$$

Using the conditions (2.1), (1.20) and (1.19a) in (2.17), we get

$$\lambda_{m_1 m_2 \ldots m_n} R^i_{jkh} + \mu_{m_1 m_2 \ldots m_n}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) + \tfrac{1}{4}\delta_{m_1 m_2 \ldots m_n}\big(R^i_k g_{jh} - R^i_h g_{jk}\big)$$
$$= \big[\, \Gamma^{*i}_{jkh} + \Gamma^{*i}_{jks} G^s_h + C^i_{jm}\big(G^m_{kh} - G^m_{kt} G^t_h\big) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh}$$
$$- \Gamma^{*i}_{jhk} - \Gamma^{*i}_{jhs} G^s_k - C^i_{jm}\big(G^m_{hk} - G^m_{ht} G^t_k\big) - \Gamma^{*i}_{mh} \Gamma^{*m}_{jk}\big]_{|m_1|m_2|\ldots|m_n}$$

By using (1.13), (1.20) and (1.19a), the above equation can be written as

$$(2.18) \qquad \big[\, \Gamma^{*i}_{jkh} + \Gamma^{*i}_{jks} G^s_h + C^i_{jm}\big(G^m_{kh} - G^m_{kt} G^t_h\big) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh} - \Gamma^{*i}_{jhk} - \Gamma^{*i}_{jhs} G^s_k$$

$$- C_{jm}^i \left(G_{hk}^m - G_{ht}^m G_k^t \right) - \Gamma_{mh}^{*i} \Gamma_{jk}^{*m} \big]_{|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} \big[\Gamma_{jkh}^{*i} + \Gamma_{jks}^{*i} G_h^s$$

$$+ C_{jm}^i \left(G_{kh}^m - G_{kt}^m G_h^t \right) + \Gamma_{mk}^{*i} \Gamma_{jh}^{*m} - \Gamma_{jhk}^{*i} + \Gamma_{jhs}^{*i} G_k^s - C_{jm}^i \left(G_{hk}^m - G_{ht}^m G_k^t \right)$$

$$- \Gamma_{mh}^{*i} \Gamma_{jk}^{*m} + \mu_{m_1 m_2 ... m_n} \big(\delta_h^i g_{jk} - \delta_k^i g_{jh} \big) + \frac{1}{4} \delta_{m_1 m_2 ... m_n} \big(R_k^i g_{jh} - R_h^i g_{jk} \big) \ .$$

Thus, we conclude

Theorem 2.5: Recurrence of Connection Parameters

In a GR^h-$n^{th} RF_n$ space, the nth-order h-covariant derivative of the tensor expression involving Cartan's connection parameters Γ_{jkh}^{*i} and associated terms is generalized n^{th}-recurrent.

Transvecting (2.18) by y^j, using (1.12b), (1.21a), (1.10b), (1.19b) and (1.9c), we get

(2.19) $\quad \left(P_{kh}^i + P_{ks}^i G_h^s + \Gamma_{mk}^{*i} G_h^m - P_{hk}^i - P_{hs}^i G_k^s - \Gamma_{mh}^{*i} G_k^m \right)_{|m_1|m_2|...|m_n}$

$$= \lambda_{m_1 m_2 ... m_n} \left(P_{kh}^i + P_{ks}^i G_h^s + \Gamma_{mk}^{*i} G_h^m - P_{hk}^i - P_{hs}^i G_k^s - \Gamma_{mh}^{*i} G_k^m \right)$$

$$+ \mu_{m_1 m_2 ... m_n} \big(\delta_h^i y_k - \delta_k^i y_h \big) + \frac{1}{4} \delta_{m_1 m_2 ... m_n} \big(R_k^i y_h - R_h^i y_k \big) \ .$$

Transvecting (2.19) by y_i, using (1.12a), (1.21b), and (1.7b), we get

(2.20) $\quad \left(-P_{jkh} - P_{jks} G_h^s + y_i \Gamma_{mk}^{*i} \Gamma_{jh}^{*m} + P_{jhk} + P_{jhs} G_k^s - y_i \Gamma_{mh}^{*i} \Gamma_{jk}^{*m} \right)_{|m_1|m_2|...|m_n}$

$$= \lambda_{m_1 m_2 ... m_n} \left(-P_{jkh} - P_{jks} G_h^s + y_i \Gamma_{mk}^{*i} \Gamma_{jh}^{*m} + P_{jhk} + P_{jhs} G_k^s - y_i \Gamma_{mh}^{*i} \Gamma_{jk}^{*m} \right)$$

$$+ \mu_{m_1 m_2 ... m_n} \big(y_h g_{jk} - y_k g_{jh} \big) + \frac{1}{4} \delta_{m_1 m_2 ... m_n} \big(R_k^i y_h - R_h^i y_k \big) y_i \ .$$

Further, transvecting (2.19) by g_{ir}, using (1.11a), (1.21c), (1.5) and (1.7b), we get

(2.21) $\quad (P_{krh} + P_{krs} G_h^s + \Gamma_{mrk}^* G_h^m - P_{hrk} - P_{hrs} G_k^s - \Gamma_{mrh}^* G_k^m)_{|m_1|m_2|...|m_n}$

$$= \lambda_{m_1 m_2 ... m_n} (P_{krh} + P_{krs} G_h^s + \Gamma_{mrk}^* G_h^m - P_{hrk} - P_{hrs} G_k^s - \Gamma_{mrh}^* G_k^m)$$

$$+ \mu_{m_1 m_2 ... m_n} (g_{hr} y_k - g_{kr} y_h) + \frac{1}{4} \delta_{m_1 m_2 ... m_n} (R_k^i y_h - R_h^i y_k) g_{ir} \ .$$

Thus, we conclude

Theorem 2.6: Recurrence of Torsion Tensors

The nth-order h-covariant derivatives of the torsion-related tensors, specifically:

1. $\left(P_{kh}^i + P_{ks}^i G_h^s + \Gamma_{mk}^{*i} G_h^m - P_{hk}^i - P_{hs}^i G_k^s - \Gamma_{mh}^{*i} G_k^m \right).$

2. $\left(-P_{jkh} - P_{jks} G_h^s + y_i \Gamma_{mk}^{*i} \Gamma_{jh}^{*m} + P_{jhk} + P_{jhs} G_k^s - y_i \Gamma_{mh}^{*i} \Gamma_{jk}^{*m} \right).$

3. $(P_{krh} + P_{krs} G_h^s + \Gamma_{mrk}^* G_h^m - P_{hrk} - P_{hrs} G_k^s - \Gamma_{mrh}^* G_k^m).$

Are governed by the generalized recurrence conditions defined in equations (2.19), (2.20), and (2.21) respectively.

3. Necessary and Sufficient Condition

In this section, we establish the fundamental relationship between Cartan's third curvature tensor R_{jkh}^i and Cartan's fourth curvature tensor K_{jkh}^i. These two tensors are connected by the following structural identity:

(3.1) $R^i_{jkh} = K^i_{jkh} + C^i_{jr} H^r_{kh}$.

By applying the h-covariant derivative of the nth order to both sides of the equation, we obtain:

(3.2) $R^i_{jkh|m_1|m_2|...|m_n} = K^i_{jkh|m_1|m_2|...|m_n} + \left(C^i_{jr} H^r_{kh}\right)_{|m_1|m_2|...|m_n}$.

Substituting the recurrence condition of the GR^h-$n^{th}RF_n$ space into this derivative, we derive the necessary and sufficient conditions for the recurrence of related curvature tensors.

Using the condition (2.1) in (3.2), we get

$$\lambda_{m_1 m_2 ... m_n} R^i_{jkh} + \mu_{m_1 m_2 ... m_n}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R^i_k\, g_{jh} - R^i_h\, g_{jk}\right)$$
$$= K^i_{jkh|m_1|m_2|...|m_n} + \left(C^i_{jr} H^r_{kh}\right)_{|m_1|m_2|...|m_n}$$.

By using the condition (3.1), the above equation implies to

(3.3) $\lambda_{m_1 m_2 ... m_n} K^i_{jkh} + \lambda_{m_1 m_2 ... m_n}\left(C^i_{jr} H^r_{kh}\right) + \mu_{m_1 m_2 ... m_n}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)$

$\quad + \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R^i_k\, g_{jh} - R^i_h\, g_{jk}\right) = K^i_{jkh|m_1|m_2|...|m_n} + \left(C^i_{jr} H^r_{kh}\right)_{|m_1|m_2|...|m_n}$

This shows that

(3.4) $K^i_{jkh|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} K^i_{jkh} + \mu_{m_1 m_2 ... m_n}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)$

$\quad + \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R^i_k\, g_{jh} - R^i_h\, g_{jk}\right)$,

if and only if

(3.5) $\left(C^i_{jr} H^r_{kh}\right)_{|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n}\left(C^i_{jr} H^r_{kh}\right)$.

Thus, we conclude

Theorem 3.1: Recurrence Condition for Cartan's Fourth Curvature Tensor

In a generalized R^h-n^{th} order recurrent Finsler space GR^h-$n^{th}RF_n$, Cartan's fourth curvature tensor K^i_{jkh} is generalized n^{th}-recurrent if and only if the tensor product $C^i_{jr} H^r_{kh}$ is recurrent of the nth order.

Contracting the indices i and h in (3.3), using (1.28a), (1.6), (1.17e) and (1.7b), we get

(3.6) $\lambda_{m_1 m_2 ... m_n} K_{jk} + \lambda_{m_1 m_2 ... m_n}\left(C^p_{jr} H^r_{kp}\right) + (n-1)\,\mu_{m_1 m_2 ... m_n} g_{jk}$

$\quad + \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R^i_k\, g_{ji} - R\, g_{jk}\right) = K_{jk|m_1|m_2|...|m_n} + \left(C^p_{jr} H^r_{kp}\right)_{|m_1|m_2|...|m_n}$.

This shows that

$$K_{jk|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n} K_{jk} + (n-1)\,\mu_{m_1 m_2 ... m_n} g_{jk}$$
$$+ \frac{1}{4}\delta_{m_1 m_2 ... m_n}\left(R^i_k\, g_{ji} - R\, g_{jk}\right) ,$$

if and only if

$$\left(C^p_{jr} H^r_{kp}\right)_{|m_1|m_2|...|m_n} = \lambda_{m_1 m_2 ... m_n}\left(C^p_{jr} H^r_{kp}\right)$$.

Thus, we conclude

Theorem 3.2: Contraction and Ricci Tensor Relations

By contracting the indices i and h in the fundamental recurrence identity, the Ricci tensor K_{jk} (associated with Cartan's fourth curvature tensor) satisfies a generalized n^{th}-order recurrence if and only if the contracted tensor $C_{jr}^p H_{kp}^r$ is n^{th}-recurrent.

Theorem 3.3: Existence of the Ricci Tensor K_{jk}

In a GR^h-$n^{th}RF_n$ space, the Ricci tensor K_{jk} is non-vanishing if and only if the tensor $C_{jr}^p H_{kp}^r$ satisfies the nth-order recurrence condition. This ensures the geometric consistency of the fourth curvature tensor within the generalized space.

Transvecting (3.6) by y^k, using (1.12b), (1.28b), (1.24) and (1.9c), we get

$$\lambda_{m_1 m_2 \dots m_n} K_j + \lambda_{m_1 m_2 \dots m_n}\left(C_{jr}^p H_p^r\right) + (n-1)\,\mu_{m_1 m_2 \dots m_n}\,y_j$$
$$+\frac{1}{4}\delta_{m_1 m_2 \dots m_n}\left(R_k^i\,g_{ji}\,y^k - R\,y_j\right) = K_{j|m_1|m_2|\dots|m_n} + \left(C_{jr}^p H_p^r\right)_{|m_1|m_2|\dots|m_n}.$$

This shows that

$$K_{j|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} K_j + (n-1)\,\mu_{m_1 m_2 \dots m_n}\,y_j$$
$$+\frac{1}{4}\delta_{m_1 m_2 \dots m_n}\left(R_k^i\,g_{ji}\,y^k - R\,y_j\right),$$

if and only if

$$\left(C_{jr}^p H_p^r\right)_{|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n}\left(C_{jr}^p H_p^r\right).$$

Thus, we conclude

Theorem 3.4: Recurrence of the Curvature Vector K_j

The curvature vector K_j associated with Cartan's fourth curvature tensor is non-vanishing in a GR^h-$n^{th}RF_n$ space if and only if the tensor $C_{jr}^p H_p^r$ is n^{th}-recurrent. This condition establishes the directional properties of the fourth curvature field in higher-order recurrent spaces.

4. Conclusion

In this study, we have successfully extended the theory of recurrent Finsler spaces by introducing the concept of Generalized n^{th}-order recurrent Finsler spaces GR^h-$n^{th}RF_n$ relative to Cartan's third curvature tensor. This research has provided a comprehensive framework that bridges the gap between lower-order recurrence properties and higher-order differential structures in Finsler geometry.

The key contributions of this paper can be summarized as follows:

1. **Framework Establishment:** We established the fundamental recurrence conditions for the nth-order h-covariant derivative of Cartan's third curvature tensor. Our analysis confirms that this generalized approach encompasses standard recurrence models as special cases, providing a more robust mathematical platform for higher-dimensional manifolds.

2. **Geometric Consistency:** Through the derivation of necessary and sufficient conditions, we proved that the fundamental geometric objects such as the Ricci tensor, the deviation tensor, and the curvature scalar remain non-vanishing in GR^h-$n^{th}RF_n$ spaces. This ensures the physical and geometric validity of the defined spaces.

3. **Relational Dynamics:** We rigorously examined the relationship between Cartan's third R^i_{jkh} and fourth K^i_{jkh} curvature tensors. We demonstrated that the recurrence of the fourth curvature tensor is intrinsically linked to the recurrence of the tensor product $C^i_{jr}H^r_{kh}$, highlighting the internal dependency of these tensors in higher-order recurrence.

4. **Special Cases:** Our findings demonstrate that while the recurrence properties hold generally, unique simplifications occur in three-dimensional and R3-like spaces, making them ideal candidates for further exploration in applied physics, particularly in theories involving gravitational field models.

This work not only advances the theoretical understanding of nth-order recurrent Finsler spaces but also paves the way for future investigations into the potential applications of these spaces in differential geometry and related fields.

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حول فضاءات فينسلر المتكررة المعممة من الرتبة n المرتبطة بموتر تقوس كارتان

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الملخص: تتناول هذه الورقة البحثية دراسة البنية الهندسية لفضاء فينسلر F_n الذي يحقق خاصية التكرار المعمم من الرتبة n بالنسبة إلى حقل موتر تقوس كارتان الثالث R^i_{jkh} . حيث نقوم بتعريف صنف جديد من فضاءات فينسلر نرمز له بالرمز $GR^{h-n^{th}}RF_n$. بحيث يحقق الاشتقاق التغايري الأفقي من الرتبة n لموتر التقوس R^i_{jkh} شرط تكرار خاص يتضمن عدة حقول لموترات التكرار. وخلال هذه الدراسة نستنتج عدة هويات هندسية أساسية، من بينها هويات بيانكي (Bianchi) وهويات من نمط فيبلن Veblen-type identities ضمن إطار هذا الفضاء المعمم. كما نبرهن الشروط اللازمة والكافية التي بموجبها يمتلك موتر التقوس الرابع لكارتان K^i_{jkh} وموتر ريتشي خاصية التكرار. وتبين النتائج أن القياس القياسي للتقوس وكذلك موتر الانحراف المرتبط بموتر التقوس الثالث لكارتان يكونان غير منعدمَين في فضاءات

$$R^{h-n^{th}}RF_n.$$

وتسهم هذه النتائج في توسيع النظرية المعروفة لفضاءات فينسلر المتكررة لتشمل المتشعبات التفاضلية ذات الرتب الأعلى.

كلمات مفتاحيه: هندسة فينسلر ؛ موترات تقوس كارتان؛ التكرار من الرتبة n ؛ الاشتقاق التغايري الأفقي؛ فضاءات $GR^{h-n^{th}}RF_n$ ؛ موتر ريتشي.