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Generalized H^h-Recurrent Finsler Spaces and Their Relations with Berwald and Cartan Curvature Tensors

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Abstract: This paper studies generalized recurrence in Finsler geometry with emphasis on the Berwald and Cartan curvature tensors. Finsler spaces extend Riemannian geometry by allowing geometric quantities to depend on both position and direction. The research introduces generalized recurrence conditions for these curvature tensors and develops theoretical relations describing their behavior. A new concept, called generalized H^h-recurrent Finsler spaces, is proposed and analyzed. The study defines these spaces through recurrence relations involving covariant vector fields, curvature tensors, and deviation tensors. Several theorems are established to characterize the recurrence properties of the Berwald tensor H^i_{jkh}. The results classify Finsler spaces according to the degree and type of recurrence. The obtained relations reveal structural connections between Berwald and Cartan tensors under generalized recurrence conditions. These findings contribute to understanding curvature structures and symmetry properties in non-Riemannian geometry. The study also extends the theoretical framework of recurrence in differential geometry and its applications in Finsler spaces.

Key words: Generalized H^h-recurrent space, Berwald curvature tensor, Finsler geometry, Covariant derivatives.

1. Introduction

Finsler geometry, a generalization of Riemannian geometry, provides a rich framework for exploring geometric properties of spaces where the length of a tangent vector is not necessarily quadratic. A particularly important class of Finsler spaces is those defined by the behavior of their curvature tensors. Among them, the concept of recurrent spaces plays a crucial role in understanding the symmetry and structure of such spaces. In this study, we introduce and analyze the concept of a generalized H^h-recurrent space in the context of Finsler geometry.

The generalized H^h-recurrent space, denoted by GH^h-RF_n , is characterized by specific recurrence conditions on the Berwald curvature tensor H^i_{jkh}, which describe how this tensor behaves under certain transformations and conditions. These conditions are expressed through a set of differential equations involving the curvature tensor and various covariant vector fields, such as λ_l , μ_l and δ_l . Our investigation into this space extends the existing theory of recurrent spaces by incorporating generalized recurrence relations that apply to more complex geometries.

In the context of affinely connected spaces, which are defined by the vanishing of certain connection parameters, we further examine how the properties of the generalized H^h-recurrent space relates to the behavior of tensors such as the H-Ricci tensor and the deviation tensor. The connection between generalized recurrent spaces and affinely connected spaces is also explored, with specific emphasis on the conditions under which the curvature tensor satisfies generalized recurrence relations.

By examining these spaces, we seek to provide new insights into the structure of Finsler spaces, particularly in relation to their curvature tensors and their potential applications in theoretical physics, such as in the study of gravitational fields and other geometric phenomena. Through a series of theorems, we establish key conditions that define when the generalized H^h-recurrent space exhibits specific geometric behaviors, thereby expanding the mathematical understanding of recurrent Finsler spaces and their applications. Finsler geometry, an extension of Riemannian geometry, has become a significant area of study in modern differential geometry, offering a broader framework for understanding curved spaces. Numerous studies have contributed to the development of Finsler spaces and their various characteristics, focusing on special curvature tensors, recurrence relations, and different geometric properties.

Al-Qashbari, Abdallah, and Al-ssallal (2024) have explored recurrent Finsler structures with higher-order generalizations, emphasizing the role of special curvature tensors in the geometry of Finsler spaces. In their subsequent work, Al-Qashbari and Al-ssallal (2024) investigated curvature tensors through Berwald's and Cartan's higher-order derivatives, further advancing the understanding of Finsler space's complex geometric structures. Moreover, their decomposition analysis of Weyl's curvature tensor via Berwald's derivatives (2024) has provided insightful contributions to the application of curvature tensor theories in Finsler geometry.

The work of Al-Qashbari (2020) has focused on the study of generalized curvature tensors and recurrence decompositions in Finsler spaces, offering a comprehensive investigation into B-recurrent Finsler spaces and curvature identities. His research, particularly on generalized BR-trirecurrent Finsler spaces (2017), continues to be a foundational reference in the field. Al-Qashbari and his collaborators, including Abdallah and Nasr (2025), furthered this work by examining generalized trirecurrent spaces in the context of Gh-covariant derivatives, highlighting the importance of this approach for both theoretical and practical aspects of Finsler geometry.

In addition to these advancements, other researchers have made significant contributions to Finsler geometry. Notably, Izumi (1976) introduced the concept of P*-Finsler space, laying the groundwork for further exploration in the field. Pandey and Misra (1981) and Pandey and Pal (2003) have explored projective recurrent and hyper-surface structures in Finsler spaces, respectively, providing valuable insights into the broader geometric properties of these spaces. Maralebhavi and Rathnamma (1999) studied generalized recurrent and concircular recurrent manifolds, expanding the scope of recurrent geometry in differential manifolds. Goswami (2017) conducted a systematic review of special types of Finsler spaces, offering a broad perspective on their application in differential geometry. The research by Pandey, Saxena, and Goswami (2011) furthered the study of generalized H-recurrent space, contributing to the classification and properties of recurrent spaces. Rund's classic work (1959) on the differential geometry of Finsler spaces remains an essential reference in the field, providing foundational concepts and techniques that have shaped the development of modern Finsler geometry. These diverse studies, spanning over several decades, have collectively enhanced the understanding of Finsler spaces, their curvature tensors, and various recurrence properties. The current research builds upon this extensive body of work, aiming to further explore the intricate relationships and higher-order generalizations within the framework of Finsler geometry.

Let us consider an n-dimensional Finsler space F_n equipped with the metric function F satisfying the requisite conditions [15]. Let the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters $\Gamma^{*i}_{jk}$and Berwald's connection parameters G^i_{jk} They are symmetric in their lower indices and positively homogeneous of degree zero in the directional arguments.

The vectors y_i , y^i and metrics g_{ij} , g^{ij} and δ^i_j satisfies the following relations

(1.1) a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $g_{ij} = \dot{\partial}_i\, y_j = \dot{\partial}_j\, y_i$,

d) $g_{ij}\, y^j = \frac{1}{2}\dot{\partial}_i F^2 = F\,\dot{\partial}_i F$, e) $\dot{\partial}_j\, y^i = \delta^i_j$, f) $\delta^i_k\, y_i = y_k$,

g) $\delta^i_k\, g_{ji} = g_{jk}$, h) $\delta^i_k\, g^{jk} = g^{ik}$, i) $\delta^i_k\, \delta^k_h = \delta^i_h$, and

j) $g_{ij}\, g^{jk} = \delta^i_k = \begin{cases} 1 & if \quad i = k \;, \\ 0 & if \quad i \neq k \end{cases}$.

The tensor C_{ijk} defined by

(1.2) $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2$.

is known as (h) hv - torsion tensor [7]. It is positively homogeneous of degree -1 in the directional arguments and symmetric in all its indices.

The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor C_{ijk} are related by

(1.3) a) $C^h_{ik} = g^{hj}C_{ijk}$ and b) $C_{ijk} = g_{hj}\, C^h_{ik}$.

The (v) hv-torsion tensor C^h_{ik} is also positively homogeneous of degree -1 in the directional arguments and symmetric in its lower indices.

É. Cartan deduced the h-covariant derivative for an arbitrary vector filed X^i with respect to x^k given by

(1.4) $X^i_{|k} = \partial_k\, X^i - (\dot{\partial}_r X^i)\, G^r_k + X^r\Gamma^{*i}_{rk}$.

The metric tensor g_{ij} and the vector y^i are covariant constant with respect to above process, i.e.

(1.5) a) $g_{ij|k} = 0$, b) $y^i_{|k} = 0$ and c) $g^{pr}{}_{|k} = 0$.

The process of h-covariant differentiation defined above commute with partial differentiation with respect to y^j for arbitrary vector filed X^i , according to

(1.6) $\dot{\partial}_j\left(X^i_{|k}\right) - (\dot{\partial}_j\, X^i)_{|k} = X^r(\dot{\partial}_j\, \Gamma^{*i}_{rk}) - (\dot{\partial}_r\, X^i)\, P^r_{jk}$, were

(1.7) a) $\dot{\partial}_j\Gamma^{*r}_{hk} = \Gamma^{*r}_{jhk}$ and b) $P^i_{kh}\, y^k = P^i_{kh}\, y^h = 0$.

The tensor P^i_{kh} is called v(hv) –torsion tensor and its associate tensor P_{jkh} is given by

(1.8) $g_{rj}\, P^r_{kh} = P_{kjh}$.

The quantities H^i_{jkh} and H^i_{kh} form the components of tensors and they called h-curvature tensor of Berwald (Berwald curvature tensor) and h(v)–torsion tensor, respectively and defined as follow:

(1.9) a) $H^i_{jkh} = \partial_j G^i_{kh} + G^r_{kh}G^i_{rj} + G^i_{rhj}G^r_k - \partial_j G^i_{hk} - G^r_{hk}G^i_{rj} - G^i_{rkj}G^r_h$,

and

(1.9) b) $H^i_{kh} = \partial_h G^i_k + G^r_k C^i_{rh} - \partial_k G^i_h - G^r_h C^i_{rk}$.

They are skew-symmetric in their lower indices, i.e. k and h. Also, they are positively homogeneous of degree zero and one, respectively in their directional arguments. They are also related by

(1.10) a) $H^i_{jkh}y^j = H^i_{kh}$, b) $H^i_{jkh} = \partial_j\, H^i_{kh}$ and c) $H^i_{jk} = \partial_j\, H^i_k$.

These tensors were constructed initially by mean of the tensor H^i_h , called the deviation tensor, given by

(1.11) $H^i_h = 2\,\partial_h G^i - \partial_r\, G^i_h\, y^r + 2G^i_{hs}\, G^s - G^i_s\, G^s_h$.

The deviation tensor H^i_h is positively homogeneous of degree two in the directional arguments.

In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (1.10) and (1.11), we have the following:

(1.12) a) $H^i_{jk}\, y^j = H^i_k = -H^i_{kj}\, y^j$, b) $H_{jk} = H^r_{jkr}$, c) $H_j = H^r_{jr}$, d) $H^r_{rkh} = H_{hk} - H_{kh}$,

e) $(n-1)H = H^r_r$, f) $y_i\, H^i_j = H^i_j\, y^j = 0$ and g) $-H_h = H^r_{rh}$.

The contracted tensor H_{kh}(Ricci tensor), H_k (Curvature vector) and H (curvature scalar) are also connected by

(1.13) a) $H_{kh} = \dot{\partial}_k\, H_h$, b) $H_{kh}\, y^k = H_h$ and c) $H_k\, y^k = (n-1)\, H$.

The quantities H^i_{jkh} and H^i_{kh} are satisfies the following

(1.14) a) $H_{ijkh} = g_{jr}H^r_{ihk}$, b) $H_{jk.h} = g_{jr}H^r_{hk}$ and c) $H^i_{jkh} + H^i_{hjk} + H^i_{khj} = 0$.

P.N. Pandey proved

(1.15) $y_i\, H^i_{hk} = 0$.

Cartan's third curvature tensor R^i_{jkh} satisfies the identity known as Bianchi identity

(1.16) a) $R^i_{jkh|s} + R^i_{jsk|h} + R^i_{jhs|k} + \left(R^r_{mhs}P^i_{jkr} + R^r_{mkh}P^i_{jsr} + R^r_{msk}P^i_{jhr}\right)y^m = 0$,

b) $R^i_{jkh}\, y^j = H^i_{kh} = K^i_{jkh}\, y^j$, c) $R_{ijhk} = g_{rj}\, R^r_{ihk}$, and

d) $R_{jkhm}\, y^j = H_{kh.m}$.

Also, this tensor satisfies the following relation too

(1.17) a) $R^i_{jkh} = K^i_{jkh} + C^i_{jm}\, H^m_{kh}$, b) $R_{ijkh} = K_{ijkh} + C_{ijs}\, H^s_{kh}$, and

c) $H^i_{jkh} = K^i_{jkh} + y^s(\dot{\partial}_j K^i_{skh})$,

where R_{ijkh} is the associate curvature tensor of R^i_{jkh}. Cartan's fourth curvature tensor K^i_{jkh} and its associate curvature tensor K_{ijkh} satisfy the following identities known as Bianchi

(1.18) a) $K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0$ and b) $K_{jrkh} + K_{hrjk} + K_{krhj} = 0$.

(1.19) a) $K_{ijkh} = g_{rj}\, K^r_{ikh}$, b) $K^i_{jki} = K_{jk}$, c) $g^{jk}\, K_{jk} = K$ and d) $g^{ik}K_{jk} = K^i_j$.

(1.20) a) $R^i_{jki} = R_{jk}$, b) $g^{jk}R_{jk} = R$ and c) $g^{ik}R_{jk} = R^i_j$.

2. A Generalized H^h-Recurrent Space

This research introduces a new definition of the generalized H^h-recurrent spaces as following:

Let us consider a Finsler space F_n whose Berwald curvature tensor H^i_{jkh} satisfies the condition

(2.1) $H^i_{jkh|l} = \lambda_l H^i_{jkh} + \mu_l\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \frac{1}{4}\delta_l\left(H^i_k\, g_{jh} - H^i_h\, g_{jk}\right)$, $H^i_{jkh} \neq 0$,

where λ_l , μ_l and δ_l are non-null covariant vectors field. We shall call such space as a generalized H^h-recurrent space. We shall denote it briefly by GH^h-RF_n .

Now, let us consider a generalized H^h- recurrent space characterized by the condition (2.1) .

Transvecting the condition (2.1) by y^j , using (1.5b), (1.10a) and (1.1a), we get

(2.2) $H^i_{kh|l} = \lambda_l H^i_{kh} + \mu_l\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) + \frac{1}{4}\delta_l\left(H^i_k\, y_h - H^i_h\, y_k\right)$.

Further, transvecting the condition (2.2) by y^k, using (1.5b), (1.12a), (1.12f) and (1.1b), we get

(2.3) $H^i_{h|l} = \lambda_l H^i_h + \mu_l\left(\delta^i_h\, F^2 - y_h\, y^i\right) + \frac{1}{4}\delta_l\left(H^i_h\, F^2\right)$.

Transvecting the condition (2.2) by g_{ip} , using (1.5a), (1.1g) and (1.14b), we get

(2.4) $H_{kp.h|l} = \lambda_l\, H_{kp.h} + \mu_l\left(g_{hp}\, y_k - g_{kp}\, y_h\right) + \frac{1}{4}\delta_l\left(H^i_k\, y_h - H^i_h\, y_k\right) g_{ip}$.

Contracting indices i and h in (2.1), (2.2) and (2.3) and using (1.12b), (1.12c), (1.1g), (1.1f), (1.1b) (1.12e) and (1.12f), we get

(2.5) $H_{jk|l} = \lambda_l H_{jk} + \mu_l(n-1)g_{jk} + \frac{1}{4}\delta_l\left(H^i_k\, g_{ji} - (n-1)H\, g_{jk}\right)$

(2.6) $H_{k|l} = \lambda_l H_k + \mu_l(n-1)y_k - \frac{1}{4}\delta_l\left((n-1)H\, y_k\right)$, and

(2.7) $H_{|l} = \lambda_l H + \mu_l\, F^2 + \frac{1}{4}\delta_l(H\, F^2)$

Consequently, we have established that

Theorem 2.1. In the generalized GH^h-RF_n space, the h-covariant derivatives of the h(v)-torsion tensor H^i_{kh} , the deviation tensor H^i_h , the tensor $H_{kp.h}$, the Ricci tensor H_{jk} , the vector tensor H_k and the scalar H are non-vanishing .

By partially differentiating equation (2.2) with respect to y^j , and employing equations (1.10b), (1.10c), (1.7a), and (1.1c), along with the commutation formula given in (1.6) for the h(v)-torsion tensor H^i_{jk} , we obtain

(2.8) $H^i_{jkh|l} + H^r_{kh}\,\Gamma^{*i}_{jrl} - H^i_{rh}\,\Gamma^{*r}_{jkl} - H^i_{kr}\,\Gamma^{*r}_{jhl} - H^i_{rkh}\, P^r_{jl} = \left(\dot{\partial}_j\lambda_l\right)H^i_{kh} + \lambda_l H^i_{jkh}$
$+ \left(\dot{\partial}_j\, \mu_l\right)\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) + \mu_l\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \frac{1}{4}\left(\dot{\partial}_j\delta_l\right)\left(H^i_k\, y_h - H^i_h\, y_k\right)$
$+ \frac{1}{4}\delta_l\left(H^i_{jk}\, y_h - H^i_{jh}\, y_k\right)$.

This shows that

$$H^i_{jkh|l} = \lambda_l H^i_{jkh} + \mu_l\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \frac{1}{4}\delta_l\left(H^i_{jk}\, y_h - H^i_{jh}\, y_k\right)$$

if and only if

(2.9) $H^r_{kh}\,\Gamma^{*i}_{jrl} - H^i_{rh}\,\Gamma^{*r}_{jkl} - H^i_{kr}\Gamma^{*r}_{jhl} - H^i_{rkh}\,P^r_{jl} = (\dot{\partial}_j\lambda_l)\,H^i_{kh} + (\dot{\partial}_j\,\mu_l)(\delta^i_h\,y_k - \delta^i_k\,y_h)$
$+\frac{1}{4}(\dot{\partial}_j\delta_l)(H^i_k\,y_h - H^i_h\,y_k)$.

Consequently, we have established that

Theorem 2.2. In the space $GH^h\text{-}RF_n$, the Berwald curvature tensor H^i_{jkh} is generalized recurrent if and only if (2.9) is satisfied.

By transvecting equation (2.8), with g_{ip} , and utilizing equations (1.7a), (1.1g), (1.14a) and (1.14b), we obtain the following result:

(2.10) $H_{jpkh|l} + g_{ip}\,(H^r_{kh}\,\Gamma^{*i}_{jrl} - H^i_{rh}\,\Gamma^{*r}_{jkl} - H^i_{kr}\,\Gamma^{*r}_{jhl} - H^i_{rkh}\,P^r_{jl}) = \lambda_l\,H_{jpkh} + (\dot{\partial}_j\,\lambda_l)\,H_{kp.h}$
$+ (\dot{\partial}_j\,\mu_l)(g_{hp}\,y_k - g_{kp}\,y_h) + \mu_l(g_{hp}\,g_{jk} - g_{kp}\,g_{jh}) + \frac{1}{4}(\dot{\partial}_j\delta_l)(H^i_k\,y_h - H^i_h\,y_k)\,g_{ip}$
$+\frac{1}{4}\delta_l(H_{jp.k}\,y_h - H_{jp.h}\,y_k)$.

This shows that

$$H_{jpkh|l} = \lambda_l\,H_{jpkh} + \mu_l\,(g_{jk}\,g_{hp} - g_{jh}\,g_{kp}) + \frac{1}{4}\delta_l(H_{jp.k}\,y_h - H_{jp.h}\,y_k) ,$$

if and only if

(2.11) $g_{ip}\,(H^r_{kh}\,\Gamma^{*i}_{jrl} - H^i_{rh}\,\Gamma^{*r}_{jkl} - H^i_{kr}\,\Gamma^{*r}_{jhl} - H^i_{rkh}\,P^r_{jl}) = (\dot{\partial}_j\,\lambda_l)\,H_{kp.h}$
$+ (\dot{\partial}_j\,\mu_l)(g_{hp}\,y_k - g_{kp}\,y_h) + \mu_l(g_{hp}\,g_{jk} - g_{kp}\,g_{jh}) + \frac{1}{4}(\dot{\partial}_j\delta_l)(H^i_k\,y_h - H^i_h\,y_k)\,g_{ip}$.

Consequently, we have established that

Theorem 2.3. In the space $GH^h\text{-}RF_n$, the associate tensor H_{jpkh} of the Berwald curvature tensor H^i_{jkh} is generalized recurrent if and only if (2.11) is satisfied.

By contracting the indices i and j in equation (2.8) and applying equation (1.12d) and (1.12g), we obtain the following result:

(2.12) $(H_{hk} - H_{kh})_{|l} + H^r_{kh}\Gamma^{*p}_{prl} - H^p_{rh}\Gamma^{*r}_{pkl} - H^p_{kr}\Gamma^{*r}_{phl} - H^p_{rkh}P^r_{pl} = (\dot{\partial}_p\lambda_l)H^p_{kh} + \lambda_l(H_{hk} - H_{kh})$
$+(\dot{\partial}_p\,\mu_l)(\delta^p_h\,y_k - \delta^p_k\,y_h) + \frac{1}{4}(\dot{\partial}_r\delta_l)(H^r_k\,y_h - H^r_h\,y_k) + \frac{1}{4}\delta_l(H_h\;\;y_k - H_k\;\;y_h)$.

This shows that

$$(H_{hk} - H_{kh})_{|l} = \lambda_l(H_{hk} - H_{kh}) + \frac{1}{4}\delta_l(H_h\;\;y_k - H^i_k\,y_h) ,$$

if and only if

(2.13) $H^r_{kh}\,\Gamma^{*p}_{prl} - H^p_{rh}\,\Gamma^{*r}_{pkl} - H^p_{kr}\,\Gamma^{*r}_{phl} - H^p_{rkh}\,P^r_{pl} = (\dot{\partial}_p\lambda_l)\,H^p_{kh}$
$+ (\dot{\partial}_p\,\mu_l)(\delta^p_h\,y_k - \delta^p_k\,y_h) + \frac{1}{4}(\dot{\partial}_r\delta_l)(H^r_k\,y_h - H^r_h\,y_k)$.

Consequently, we have established that

Theorem 2.4. In the space $GH^h\text{-}RF_n$, the tensor $(H_{hk} - H_{kh})$ is generalized recurrent if and only if (2.13) holds.

Next, by partially differentiating condition (2.3) with respect to y^k, and utilizing equations (1.10c) and (1.1c) together with the commutation formula given in equation (1.6) for the deviation tensor H^i_h, we derive the following result:

(2.14) $H^i_{kh|l} + H^r_h\,\Gamma^{*i}_{krl} - H^i_r\,\Gamma^{*r}_{khl} - H^i_{rh}\,P^r_{kl} = (\dot{\partial}_k\lambda_l)H^i_h + \lambda_l\,H^i_{kh} + (\dot{\partial}_k\mu_l)(\delta^i_h F^2 - y_h\,y^i)$
$+ \mu_l\,(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}(\dot{\partial}_k\delta_l)(H^i_h\,F^2) + \frac{1}{4}\delta_l(H^i_{kh}\,F^2)$.

By interchanging the indices k and h in equation (2.14), subtracting the resulting equation from (2.14), and utilizing equation (1.12c), we obtain the following result:

(2.15) $(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)_{|l} + (H^r_h\,\Gamma^{*i}_{krl} - H^i_r\,\Gamma^{*r}_{khl} - H^i_{rh}\,P^r_{kl} - k/h) = \lambda_l(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$
$- 2\,\mu_l(\delta^i_k\,g_{jh}) + [(\dot{\partial}_k\,\lambda_l)H^i_h + (\dot{\partial}_k\,\mu_l)(\delta^i_h\,F^2 - y_h\,y^i) - k/h]$
$+\frac{1}{4}(\dot{\partial}_k\delta_l)(H^i_h\,F^2) - \frac{1}{4}(\dot{\partial}_h\delta_l)(H^i_k\,F^2) + \frac{1}{4}\delta_l(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)\,F^2$.

This shows that

$$(\dot{\partial}_k\,H^i_h - \dot{\partial}_h\,H^i_k)_{|l} = \lambda_l(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k) + \frac{1}{4}\delta_l(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)\,F^2 ,$$

if and only if

(2.16) $\left(H_h^r\,\Gamma_{krl}^{*i} - H_r^i\,\Gamma_{khl}^{*r} - H_{rh}^i\,P_{kl}^r - k/h\right) = \left[\left(\dot\partial_k\,\lambda_l\right)H_h^i + \left(\dot\partial_k\,\mu_l\right)\left(\delta_h^i\,F^2 - y_h\,y^i\right) - k/h\right]$
$- 2\,\mu_l\left(\delta_k^i\,g_{jh}\right) + \frac{1}{4}\left(\dot\partial_k\delta_l\right)\left(H_h^i\,F^2\right) - \frac{1}{4}\left(\dot\partial_h\delta_l\right)\left(H_k^i\,F^2\right)$.

Consequently, we have established that

Theorem 2.5. In the space $GH^h\text{-}RF_n$, the tensor $\left(\dot\partial_k H_h^i - \dot\partial_h H_k^i\right)$ is generalized recurrent if and only if (2.16) holds.

3. Cartan's Third and Fourth Curvature Tensors in Generalized H^h-Recurrent Space

We introduce the relationship between the Berwald curvature tensor and third and fourth Cartan's curvature tensors in the space $GH^h\text{-}RF_n$.

Differentiating (1.17c) covariantly with respect to x^l in the sense of Cartan and using (2.1) and (1.17c), we get

(3.1) $K_{jkh|\ell}^i = \lambda_l K_{jkh}^i + \mu_l\left(\delta_h^i\,g_{jk} - \delta_k^i\,g_{jh}\right) + \frac{1}{4}\delta_l\left(H_k^i\,g_{jh} - H_h^i\,g_{jk}\right) + \lambda_l y^s\left(\dot\partial_j K_{skh}^i\right) - y^s\left(\dot\partial_j K_{skh}^i\right)_{|\ell}$.

This shows that

$$K_{jkh|\ell}^i = \lambda_l K_{jkh}^i + \mu_l\left(\delta_h^i\,g_{jk} - \delta_k^i\,g_{jh}\right) + \frac{1}{4}\delta_l\left(H_k^i\,g_{jh} - H_h^i\,g_{jk}\right) ,$$

if and only if

(3.2) $y^s\left(\dot\partial_j K_{skh}^i\right)_{|\ell} = \lambda_l y^s\left(\dot\partial_j K_{skh}^i\right)$.

Transvecting equation (3.1), with g_{ip} , and utilizing equations (1.5a), (1.19a), we obtain the following result

(3.3) $K_{jpkh|l} = \lambda_l\,K_{jpkh} + \mu_l\left(g_{jk}\,g_{hp} - g_{jh}\,g_{kp}\right) + \frac{1}{4}\delta_l\left(H_{jpk}\,y_h - H_{jph}\,y_k\right)$
$+ \lambda_l y^s\left(\dot\partial_j\,K_{spkh}\right) - y^s\left(\dot\partial_j\,K_{spkh}\right)_{|\ell}$.

This shows that

$$K_{jpkh|l} = \lambda_l\,K_{jpkh} + \mu_l\left(g_{jk}\,g_{hp} - g_{jh}\,g_{kp}\right) + \frac{1}{4}\delta_l\left(H_{jpk}\,y_h - H_{jph}\,y_k\right) ,$$

if and only if

(3.4) $y^s\left(\dot\partial_j\,K_{spkh}\right)_{|\ell} = \lambda_l y^s\left(\dot\partial_j\,K_{spkh}\right)$.

Contracting the indices i and h in equation (3.1) and using (1.19b), (1.1g), (1.1j) and (1.12e), we get

(3.5) $K_{jk|l} = \lambda_l K_{jk} + \mu_l(n-1)g_{jk} + \frac{1}{4}\delta_l\left(H_k^i\,g_{ji} - (n-1)H\,g_{jk}\right) + \lambda_l y^s\left(\dot\partial_j K_{sk}\right) - y^s\left(\dot\partial_j K_{sk}\right)_{|\ell}$.

This shows that

$$K_{jk|l} = \lambda_l K_{jk} + (n-1)\mu_l g_{jk} + \frac{1}{4}\delta_l\left(H_k^i\,g_{ji} - (n-1)H\,g_{jk}\right) ,$$

if and only if

(3.6) $y^s\left(\dot\partial_j K_{sk}\right)_{|\ell} = \lambda_l y^s\left(\dot\partial_j K_{sk}\right)$.

Transvecting equation (3.5), with g^{jk} , using (1.19c), (1.1j) , (1.5c) and (1.19d), we get

(3.7) $K_{|l} = \lambda_l K + (n-1)\mu_l - \frac{1}{4}(n-1)\delta_l H + \lambda_l y^s\left(\dot\partial_j K_s^j\right) - y^s\left(\dot\partial_j K_s^j\right)_{|\ell}$.

This shows that

$$K_{|l} = \lambda_l K + (n-1)\mu_l - \frac{1}{4}(n-1)\delta_l H ,$$

if and only if

(3.8) $y^s\left(\dot\partial_j K_s^j\right)_{|\ell} = \lambda_l y^s\left(\dot\partial_j K_s^j\right)$.

Transvecting equation (3.5), with g^{ik} , using (1.19d), (1.5c) and (1.1j), we get

(3.9) $K_{j|\ell}^i = \lambda_l K_j^i + (n-1)\,\mu_l\delta_j^i - \frac{1}{4}\delta_l\left((n-1)H\,\delta_j^i\right) + \lambda_l y^s\left(\dot\partial_j K_s^i\right) - y^s\left(\dot\partial_j K_s^i\right)_{|\ell}$

This shows that

$$K_{j|\ell}^i = \lambda_l K_j^i + (n-1)\,\mu_l\delta_j^i - \frac{1}{4}\delta_l\left((n-1)H\,\delta_j^i\right) ,$$

if and only if

(3.10) $y^s\left(\dot\partial_j K_s^i\right)_{|\ell} = \lambda_l y^s\left(\dot\partial_j K_s^i\right)$.

Thus, we have

Theorem3.1. In the space GH^h-RF_n , Cartan's fourth curvature tensor K^i_{jkh} , the associate tensor K_{jpkh} of Cartan's fourth curvature tensor K^i_{jkh} , the Ricci tensor K_{jk} , the scalar curvature K and the deviation tensor K^i_j are satisfying the generalized recurrent property if and only if the conditions (3.2), (3.4), (3.6), (3.8) and (3.10) are satisfied the recurrent property .

Differentiating (1.17a) covariantly with respect to x^l in the sense of Cartan, we get

(3.11) $R^{\ i}_{jkh|\ell} = K^{\ i}_{jkh|\ell} + \ (C^{\ i}_{jm} H^{\ m}_{kh})_{|\ell}$.

Using (3.1) and (1.17a) in (3.11), we get

(3.12) $R^i_{jkh|\ell} = \lambda_l R^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) + \frac{1}{4}\delta_l(H^i_k\, g_{jh} - H^i_h\, g_{jk})$
$+ \lambda_l y^s(\dot\partial_j K^i_{skh}) - y^s(\dot\partial_j K^i_{skh})_{|\ell} + (C^{\ i}_{jm} H^{\ m}_{kh})_{|\ell} - \lambda_l(C^{\ i}_{jm} H^{\ m}_{kh})$.

This shows that

$$R^i_{jkh|\ell} = \lambda_l R^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) + \frac{1}{4}\delta_l(H^i_k\, g_{jh} - H^i_h\, g_{jk}) \ ,$$

if and only if

(3.13) $y^s(\dot\partial_j K^i_{skh})_{|\ell} = \lambda_l y^s(\dot\partial_j K^i_{skh})$, and
$(C^{\ i}_{jm} H^{\ m}_{kh})_{|\ell} = \ \lambda_l(C^{\ i}_{jm} H^{\ m}_{kh})$

Transvecting equation (3.12), with g_{ip} , and utilizing equations (1.5a), (1.1g), (1.3b) and (1.16c), we obtain the following result:

(3.14) $R_{jpkh|l} = \lambda_l\, R_{jpkh} + \mu_l\,(g_{jk}\, g_{hp} - g_{jh}\, g_{kp}) + \frac{1}{4}\delta_l(H_{jpk}\, y_h - H_{jph}\, y_k)$
$\lambda_l y^s(\dot\partial_j\, K_{spkh}) - y^s(\dot\partial_j\, K_{spkh})_{|\ell} + (C_{jpm} H^{\ m}_{kh})_{|\ell} - \lambda_l(C_{jpm} H^{\ m}_{kh})$

This shows that

$$R_{jpkh|l} = \lambda_l\, R_{jpkh} + \mu_l\,(g_{jk}\, g_{hp} - g_{jh}\, g_{kp}) + \frac{1}{4}\delta_l(H_{jpk}\, y_h - H_{jph}\, y_k) \ ,$$

if and only if

(3.15) $y^s(\dot\partial_j\, K_{spkh})_{|\ell} = \ \lambda_l y^s(\dot\partial_j\, K_{spkh})$, and
$(C_{jpm} H^{\ m}_{kh})_{|\ell} = \ \lambda_l(C_{jpm} H^{\ m}_{kh})$

Contracting the indices i and h in equation (3.12) and using (1.1j), (1.19b), (1.12e) and (1.20a), we get

(3.16) $R_{jk|l} = \lambda_l R_{jk} + (n-1)\mu_l g_{jk} + \frac{1}{4}\delta_l(H^i_k\, g_{ji} - (n-1)H\, g_{jk})$
$+\lambda_l y^s(\dot\partial_j K_{sk}) - y^s(\dot\partial_j K_{sk})_{|\ell} + (C^{\ i}_{jm} H^{\ m}_{ki})_{|\ell} - \lambda_l(C^{\ i}_{jm} H^{\ m}_{ki})$.

This shows that

$$R_{jk|l} = \lambda_l R_{jk} + (n-1)\mu_l g_{jk} + \frac{1}{4}\delta_l(H^i_k\, g_{ji} - (n-1)H\, g_{jk}) \ ,$$

if and only if

(3.17) $y^s(\dot\partial_j K_{sk})_{|\ell} = \ \lambda_l y^s(\dot\partial_j K_{sk})$, and
$(C^{\ i}_{jm} H^{\ m}_{ki})_{|\ell} = \ \lambda_l(C^{\ i}_{jm} H^{\ m}_{ki})$.

Transvecting equation (3.16), with g^{jk} , using (1.20b), (1.12a) and (1.5c), (1.1j) and (1.19d), we get

(3.18) $R_{|l} = \lambda_l R + (n-1)\mu_l - \frac{1}{4}(n-1)\delta_l H + \lambda_l y^s(\dot\partial_j K^j_s) - y^s(\dot\partial_j K^j_s)_{|\ell} + (C^{\ i}_m H^{\ m}_i)_{|\ell} - \lambda_l(C^{\ i}_m H^{\ m}_i)$.

This shows that

$$R_{|l} = \lambda_l R + (n-1)\mu_l - \frac{1}{4}(n-1)\delta_l H \ ,$$

if and only if

(3.19) $y^s(\dot\partial_j K^j_s)_{|\ell} = \lambda_l y^s(\dot\partial_j K^j_s)$, and
$(C^{\ i}_m H^{\ m}_i)_{|\ell} = \lambda_l(C^{\ i}_m H^{\ m}_i)$.

Transvecting equation (3.16), with g^{ik} , using (1.120c), (1.12f), (1.19d) and (1.5c), we get

(3.20) $R^i_{j|\ell} = \lambda_l R^i_j + (n-1)\mu_l \delta^i_j - \frac{1}{4}\delta_l\left((n-1)H\, \delta^i_j\right) + \lambda_l y^s(\dot\partial_j K^i_s) - y^s(\dot\partial_j K^i_s)_{|\ell}$.

This shows that

$$R^i_{j|\ell} = \lambda_l R^i_j + (n-1)\mu_l \delta^i_j - \frac{1}{4}\delta_l\left((n-1)H\,\delta^i_j\right) ,$$

if and only if

(3.21) $y^s(\dot{\partial}_j K^i_s)_{|\ell} = \lambda_l y^s(\dot{\partial}_j K^i_s)$.

Thus, we have

Theorem 3.2. In the space GH^h-RF_n , Cartan's third curvature tensor R^i_{jkh} , the associate tensor R_{jpkh} of Cartan's third curvature tensor R^i_{jkh} , the Ricci tensor R_{jk} , the scalar curvature R and the deviation tensor R^i_j are satisfying the generalized recurrent property if and only if the conditions (3.13),(3.15),(3.17),(3.19) and (3.21) are satisfied the recurrent property **.**

4. Conclusions

In this paper, we have introduced and studied a generalized H^h-recurrent space, denoted as GH^h-RF_n , which is characterized by a specific condition involving the Berwald curvature tensor. We examined the properties and implications of the generalized recurrence of the Berwald curvature tensor and its associated tensors within such a space. Through various tensorial equations and their respective derivations, we have established key results that describe the behavior of these tensors in terms of their recurrence conditions.

The major conclusions of this study are as follows:

1. Generalized Recurrence of the Berwald Curvature Tensor: We have demonstrated that the Berwald curvature tensor H^i_{jkh} in a generalized GH^h-RF_n space is recurrent if and only if certain commutation conditions, as specified in (2.9), hold true. This recurrence property is fundamental in understanding the geometric structure of such spaces.
2. Relationship between Berwald curvature tensor H^i_{jkh} and third and fourth Cartan's curvature tensors and its associated tensors.
3. H-Ricci Tensor and H-Torsion Tensor Behavior: The recurrence properties of the H-Ricci tensor H_{jk} , the deviation tensor H^i_h , and the H-torsion tensor were studied in detail. We concluded that these tensors exhibit generalized recurrence under specific conditions, such as that given by equation (2.13). This finding contributes to a deeper understanding of the behavior of these tensors in Finsler geometry.
4. Impact of Covariant Derivatives: The analysis of the covariant derivatives of the H-tensors and their relation to the recurrence conditions has revealed important insights into the structure of generalized H^h-recurrent spaces. Specifically, the recurrence of the H-tensors is governed by the directional derivatives of the covariant vectors and the structure of the connection parameters.
5. Further Implications: This work opens avenues for further research into the geometric properties of generalized H^h-recurrent spaces, particularly in curvature tensors, and torsion properties. The established conditions may serve as the foundation for exploring more specific cases and their applications in theoretical physics, especially in the study of space-time geometry and general relativity.

In conclusion, the study of generalized H^h-recurrent spaces provide valuable insights into the structural properties of curvature tensors in Finsler geometry. The recurrence conditions derived in this work will play a crucial role in advancing the understanding of such spaces and their applications in various fields of mathematics and physics.

Potential Applications and Interdisciplinary Relevance

Although the present study is primarily theoretical, the generalized recurrence conditions established for the Berwald and Cartan curvature tensors may have potential implications beyond pure mathematics. In theoretical physics, these structures could be useful in the study of anisotropic space-time models and generalized gravitational theories, where curvature recurrence properties play a role in describing field symmetries. Moreover, the tensorial relations derived in this work could be adapted to computational frameworks in computer science, particularly in areas such as geometric modeling, optimization on manifolds, and image processing techniques that rely on curvature-based invariants. Thus, the results of this research may serve as a bridge between abstract Finsler geometry and applied disciplines, opening pathways for future interdisciplinary investigations.

References

1. Al-Qashbari, A. M., Abdallah, A. A., & Al-ssallal, F. A. (2024). Recurrent Finsler structures with higher-order generalizations defined by special curvature tensors. International Journal of Advanced Research in Science, Communication and Technology, 4(1), 68–75. DOI: 10.48175/IJARSCT-15185.
2. Al-Qashbari, A. M., & Al-ssallal, F. A. (2024). Study of curvature tensor by using Berwald's and Cartan's higher-order derivatives in Finsler space. Technological Applied and Humanitarian Academic Journal, 1(1), 1–15.
3. Al-Qashbari, A. M., Haoues, M., & Al-ssallal, F. A. (2024). A decomposition analysis of Weyl's curvature tensor via Berwald's first and second-order derivatives in Finsler spaces. Journal of Innovative Applied Mathematics and Computational Science, 4(2), 201–213. DOI: 10.46323/jiamcs. v4i2.215.
4. Al-Qashbari, A. M., Abdallah, A. A., & Baleedi, S. M. (2025). Berwald covariant derivative and Lie derivative of conharmonic curvature tensors in generalized fifth recurrent Finsler space. GPH-International Journal of Mathematics, 8(1), 24–32.
5. Al-Qashbari, A. M. A., & Qasem, F. Y. A. (2017). Study on generalized BR-trirecurrent Finsler Space. Journal of Yemen Engineer, 15, 79–89.
6. Al-Qashbari, A. M. A. (2020). On generalized curvature tensors P^i_{jkh} of second order in Finsler space. University of Aden Journal of Natural and Applied Sciences, 24(1), 171–176.
7. Al-Qashbari, A. M. A. (2019). Some properties for Weyl's projective curvature tensors of generalized Wh-birecurrent in Finsler spaces. University of Aden Journal of Natural and Applied Sciences, 23(1), 181–189.
8. Al-Qashbari, A. M. A. (2020). Some identities for generalized curvature tensors in B-recurrent Finsler space. Journal of New Theory, 32, 30–39. DOI: 10.17350/HJNP.2020.178.
9. Al-Qashbari, A. M. A. (2020). Recurrence decompositions in Finsler space. Journal of Mathematical Analysis and Modeling, 1, 77–86.
10. Al-Qashbari, A. M. A., & Al-Maisary, A. A. S. (2023). Study on generalized W^i_{jkh} of fourth-order recurrent in Finsler space. Journal of Yemeni Engineer, 17(2), 72–86.
11. Al-Qashbari, A. M. A., Abdallah, A. A., & Nasr, K. S. (2025). On generalized trirecurrent space by using Gh-covariant derivative in Finsler geometry. Journal of Mathematical Problems, Equations and Statistics, 6(1), 91–100.
12. Al-Qashbari, A. M. A., Saleh, S., & Ibedou, I. (2024). On some relations of R-projective curvature tensor in recurrent Finsler space. Journal of Non-Linear Modeling and Analysis, 6(4), 1216–1227. DOI: 10.12150/jnma.2024.1216.
13. Al-Qashbari, A. M. A., Abdallah, A. A., & Baleedi, S. M. (2025). A study of M-projective curvature tensor $\bar{W}^i_{jkh}$in $\mathrm{GBK}-5\mathrm{RF}_n$ via Lie derivative. International Journal of Applied Science and Mathematical Theory, 11(1), 1–9.
14. Al-Qashbari, A. M. A., Abdallah, A. A., & Nasr, K. S. (2025). Generalized h-torsion and curvature structures in generalized recurrent space of third order by Cartan covariant derivatives. GPH-International Journal of Mathematics, 8(2), 1–9.
15. Al-Qashbari, A. M. A., & Qasem M. M. (2026). Computational Extensions of Generalized R^h-Recurrent Finsler Spaces and Applications to Geometric Machine Learning, Journal of Science and Technology, Vol.31, No 1, 50-57.
16. Al-Qashbari, A. M. A., & Hadi W. H. A. (2025). Generalized Trirecurrence of Cartan's Second
17. Curvature Tensor in Ph-Recurrent Spaces, Journal of science and technology, Vol.30, Issue.10, 59-65.
18. Al-Qashbari, A. M. A., & AL-Ssallal F. A. M. (2025). A Study of the M-Projective Curvature Tensor $\bar{W}^i_{jkh}$ in Generalized Recurrent and Birecurrent Finsler Space, Journal of Science and Technology, Vol.30, Issue (6),16-25.

19. Al-Qashbari, A. M. A., & Mubark A. O. A. (2025). Decomposition of Generalized Recurrent Tensor Fields of R^h-nth Order in Finsler Manifolds, Journal of Science and Technology, Vol.30, Issue 2, 99-105.
20. Al-Qashbari, A. M., et al. (2026). Investigating the relationships between Weyl's and Cartan's 2th curvature tensors in Finsler spaces. International Journal of Analysis and Applications, 24, 59. https://doi.org/10.28924/2291-8639-24-59.
21. Al-Qashbari, A. M., Abdallah, A. A., & Ahmed, F. A. (2024). On generalized birecurrent Finsler space of mixed covariant derivatives in Cartan sense. International Journal of Research Publication and Reviews, 5(8), 2834–2840. https://doi.org/10.55248/gengpi.5.0824.1843.
22. Antonelli, P. L., Ingarden, R. S., & Matsumoto, M. (1993). *The theory of sprays and Finsler spaces with applications in physics and biology*. Kluwer Academic Publishers.
23. Bao, D., Chern, S. S., & Shen, Z. (2000). *An introduction to Riemann-Finsler geometry*. Springer.
24. Chern, S. S., & Shen, Z. (2005). *Riemann-Finsler geometry*. World Scientific.
25. Matsumoto, M. (1972). *Foundations of Finsler geometry and special Finsler spaces*. Kaiseisha Press.
26. Rund, H. (1959). The Differential Geometry of Finsler Space. Springer-Verlag, DOI: 10.1007/978-3-642-53099-9.

فضاءات فنسلر ذات التكرار المعمَّم H^h وعلاقاتها مع موترات الانحناء لبروفالد وكارتان

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الملخص: تتناول هذه الورقة دراسة التكرار المعمَّم في هندسة فنسلر مع التركيز على موتري الانحناء لبرفالد وكارتان. وتمثّل فضاءات فنسلر امتدادًا طبيعيًا للهندسة الريمانية، إذ تسمح بأن تعتمد الكميات الهندسية ليس فقط على الموضع بل أيضًا على الاتجاه. يقدّم البحث شروطًا معمَّمة للتكرار لهذه الموترات الانحنائية، ويطوّر علاقات نظرية تصف سلوكها. كما يقترح مفهومًا جديدًا يُعرف بفضاءات فنسلر ذات التكرار المعمَّم H^h، ويتم تحليل خصائصه. وتُعرَّف هذه الفضاءات من خلال علاقات تكرار تتضمن الحقول المتجهية التوافقية، وموترات الانحناء، وموترات الانحراف. كما تُبرهن عدة مبرهنات لوصف خصائص التكرار لموتر انحناء بروفالد H^i_{jkh}. وتُصنَّف فضاءات فنسلر وفقًا لدرجة ونوع التكرار. وتكشف العلاقات المتحصَّل عليها عن ترابطات بنيوية بين موتري بروفالد وكارتان في ظل شروط التكرار المعمَّم. وتسهم هذه النتائج في تعميق فهم بُنى الانحناء وخصائص التناظر في الهندسة غير الريمانية، كما توسّع الإطار النظري لمفهوم التكرار في الهندسة التفاضلية وتطبيقاته في فضاءات فنسلر.

كلمات مفتاحيه: الفضاء ذو التكرار المعمَّم H^h، موتر انحناء بروفالد، هندسة فنسلر، المشتقات التوافقية.