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On Generalized $\mathcal{B}$P-Recurrent Finsler Space of The Third Order

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Abstract. In this paper, we introduce a Finsler space which Cartan's second curvature tensor P^i_{jkh} satisfies the generalized trirecurrent property in sense of Berwald's, this space characterized by the following condition

$$\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P^i_{jkh} = a_{lmn} P^i_{jkh} + b_{lmn} (\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2 c_{lm} \mathcal{B}_r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) y^r$$
$$-2 d_{ln} \mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}) y^r - 2 \mu_l \mathcal{B}_n \mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}) y^r, P^i_{jkh} \neq 0 ,$$

Where $\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l$ is Berwald's covariant differential operator of third order with respect to x^l , x^m and x^n , successively. $\mathcal{B}_r$ is Berwald's covariant differential operator of first order with respect to x^r, $a_{\ell mn}$ and $b_{\ell mn}$ are non-zero covariant tensor fields of third order, $c_{\ell m}$ and $d_{\ell n}$ are non-zero covariant tensor fields of second order, μ_l is non-zero covariant vector fields and C_{jkn} is (h)-torsion tensor, such space is called as a generalized P^h-trirecurrent Finsler space and we denote by $GBP\text{-}TIR\text{-}RF_n$. we obtained the necessary and sufficient condition for some tensors to be generalized trirecurrent space.

Key words: Cartan’s second curvature tensor, Generalized $\mathcal{B}P$-trirecurrent space, Berwald’s covariant derivative of third order.

1. INTRODUCTION.

The generalized recurrent space characterized by different curvature tensors and used the sense of Berwald studied, [2], [4], [6], [9], [10], [11], [14] and [15]. The generalized birecurrent Finsler space studied in ([5], [12] and [13]). The concept of the recurrent for different curvature tensors have been discussed by F.Y.A. Qasem [7]. He studied the generalized birecurrent of first and second kind, also studied the special birecurrent of first and second kind and W^i_{jkh} generalized birecurrent Finsler space studied by F.Y.A. Qasem and A.A.M. Saleem [8] and others.

Consider a n-dimensional Finsler space, equipped with the metric function F satisfies the requisite conditions [16]. Consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ_{jk}^{*i} and Berwald's connection parameters G_{jk}^{i}. These are symmetric in their lower indices.

The vectors y_i and y^i satisfy the following relations, [16]

(a) $y_i = g_{ij}\, y^j$ and (b) $y_i\, y^i = F^2$. (1.1)

The (v)hv-torsion tensor C_{jk}^{i} and its associate (h)-hv-torsion tensor C_{jik} are related by

(a) $C_{jk}^{i}\, y^j = 0 = C_{kj}^{i}\, y^j$,

(b) $y_i C_{jk}^{i} = 0$, (c) $C_{jik} y^j = C_{ikj} y^j = 0$

and (d) $\delta_j^i C_{ikh} = C_{jkh}$. (1.2)

Berwald's covariant derivative $\mathcal{B}_k T_j^i$ of a arbitrary tensor field T_j^i with respect to x^k is given by

$$\mathcal{B}_k T_j^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i)\, G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r \ . \quad (1.3)$$

Berwald's covariant derivative of the metric function F and the vector y^i vanishes identically

(a) $\mathcal{B}_k\, y^i = 0$ and (b) $\mathcal{B}_k\, F = 0$. (1.4)

But Berwald's covariant derivative of the metric tensor g_{ij} doesn't vanish, i.e.

$\mathcal{B}_k\, g_{ij} \neq 0$.

$$\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2 y^h\, \mathcal{B}_h C_{ijk} \ . \quad (1.5)$$

Berwald's covariant differential operator with respect to x^h commutes with partial differential operator with respect to y^k, according to [16] and is given as

$$(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k) T_j^i = T_j^r G_{khr}^i - T_r^i G_{khj}^r \ , \quad (1.6)$$

where T_j^i is any arbitrary tensor field.

The curvature tensor P_{jkh}^i and the torsion tensor P_{kh}^i, satisfy the following relations:

(a) $P_{jkh}^i\, y^j = P_{kh}^i$, (b) $P_{jki}^i = P_{jk}$,

(c) $P_{jk}^i\, y^j = 0$, (d) $P_{ki}^i = P_k$,

(e) $P_k\, y^k = P$, (f) $P_i^i = P$,

(g) $g_{ir}\, P_{jkh}^r = P_{jikh}$ and

(h) $g_{ir}\, P_{kh}^r = P_{kih}$. (1.7)

The quantities H_{jkh}^i and H_{kh}^i form the components of tensors and they are called curvature tensor The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by

$$g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 \ , & if \quad i = k \ , \\ 0 \ , & if \quad i \neq k \ . \end{cases} \quad (1.8)$$

Cartan's third curvature tensor R_{jkh}^i, Cartan's fourth curvature tensor K_{jkh}^i, its associate curvature tensor K_{ijkh} and R-Ricci tensor R_{jk} in sense of Cartan, respectively, are given by [16]:

(a) $R_{jkh}^i\, y^j = H_{kh}^i = K_{jkh}^i\, y^j$,

(b) $R_{jk}\, y^j = H_k$, (c) $R_{jk}\, y^k = R_j$,

(d) $R_{jki}^i = R_{jk}$, (e) $G_k^i = \Gamma_{sk}^{*i}\, y^s$ and

(f) $H_{jk}^i\, y^k = H_j^i = -H_{kj}^i\, y^k$. (1.9)

The contracted tensor of the K-Ricci tensor K_{jk} and curvature vector K_k are connected by

(a) $K_{jki}^i = K_{jk}$, (b) $K_{jk}\, g^{jk} = K$ and

(c) $K_{jk} y^k = K_j$. (1.10)

The vector y_i and δ_k^i also satisfy the following relations:

(a) $\delta^i_k\, y_i = y_k$, (b) $\delta^i_k\, y^k = y^i$,

(c) $\delta^i_k\, g_{ji} = g_{jk}$, (d) $\delta^i_j\, g^{jk} = g^{ik}$,

(e) $\delta^i_k\, \delta^k_h = \delta^i_h$ and

(f) $g_{jh}\, y^j = y_h$. (1.11)

It is known that the Cartan's first curvature tensor S^i_{jkh} and Cartan's second curvature tensor P^i_{jkh} are connected by the following formula

$$P^i_{jkh} - P^i_{jhk} = -\, S^i_{jkh|r}\, y^r \ . \qquad (1.12)$$

We know that Cartan's fourth curvature tensor K^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} are connected by the formula:

$$R^i_{jkh} = K^i_{jkh} + C^i_{jr} H^r_{hk} \ . \qquad (1.13)$$

R-Ricci tensor R_{jk} given by

$$R_{jk} = K_{jk} + C^i_{jr} H^r_{hk} \ . \qquad (1.14)$$

2. ON GENERALIZED $\mathcal{BP}$-TRIRECURRENT SPACE

A Finsler space F_n , whose Cartan's second curvature tensor P^i_{jkh} satisfies the condition[1]

$$\mathcal{B}_l P^i_{jkh} = \lambda_l P^i_{jkh} + \mu_l\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right), \quad (2.1)$$

where $P^i_{jkh} \neq 0$ and $\mathcal{B}_l$ is covariant derivative of first order (Berwald's covariant differential operator) with respect to x^l, the quantities λ_l and μ_l are non-null covariant tensors field. It's space is called as a generalized $\mathcal{B}P$-recurrent space and is denoted briefly by $G\mathcal{B}P\text{-}RF_n$.

Consider a Finsler space F_n , whose Cartan's second curvature tensor P^i_{jkh} satisfies the following condition [2]:

$$\mathcal{B}_m\mathcal{B}_l P^i_{jkh} = u_{lm} P^i_{jkh} + v_{lm}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$-\, 2\mu_l \mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^r , \qquad (2.2)$$

where $\mathcal{B}_m\mathcal{B}_l$ is covariant derivative of second order in sense of Berwald's with respect to x^l and x^m, respectively, the quantities u_{lm} and v_{lm} are non-null covariant tensors field. They are called as a generalized $\mathcal{B}P$-birecurrent space, and is denoted briefly by $G\mathcal{B}P\text{-}BRF_n$.

To summarize above discussion note the following:

Result 1. Every generalized $\mathcal{B}P$-recurrent space is generalized $\mathcal{B}P$-birecurrent space.

Taking Berwald's covariant derivative for equation (2.2), with respect to x^n, yields

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P^i_{jkh} = (\mathcal{B}_n u_{lm}) P^i_{jkh} + u_{lm}\left(\mathcal{B}_n P^i_{jkh}\right)$$
$$+\, (\mathcal{B}_n v_{lm})\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$-\, 2 v_{lm} \mathcal{B}_r\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) y^r$$
$$-\, 2(\mathcal{B}_n \mu_l) \mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^r$$
$$-\, 2\mu_l\, \mathcal{B}_n\mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^r.$$

In view equation (2.1), the above equation is reduced to

$$\mathcal{B}_n\,\mathcal{B}_m\mathcal{B}_l\, P^i_{jkh} = a_{lmn}\, P^i_{jkh}$$
$$+\, b_{lmn}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)$$
$$-\, 2\, c_{lm}\, \mathcal{B}_r\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) y^r$$
$$-\, 2 d_{ln}\, \mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^r$$
$$-\, 2\, \mu_l\, \mathcal{B}_n\mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^r , \qquad (2.3)$$

where $a_{lmn} = \mathcal{B}_n u_{lm} + u_{lm}\, \lambda_n$, $b_{lmn} = \mathcal{B}_n v_{lm} + u_{lm}\, \mu_n$, $c_{lm} = v_{lm}$, $d_{ln} = \mathcal{B}_n \mu_l$ and $\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l$ is covariant derivative of third order in sense of Berwald's with respect to x^l , x^m and x^n , successively.

Definition 2.1. A Finsler space F_n , whose Cartan's second curvature tensor P^i_{jkh} satisfies the condition (2.3), where a_{lmn} and b_{lmn} are non-null covariant tensors field, is called a generalized $\mathcal{B}R$-trirecurrent space. It is denoted by G$\mathcal{B}$P-TIRRF$_n$.

Result 2. In generalized $\mathcal{B}$-recurrent space, every generalized $\mathcal{B}R$-birecurrent space is G$\mathcal{B}$P-TIRRF$_n$.

Transvecting the condition (2.3) by y^j, using equations (1.1a), (1.2c), (1.4a), (1.7a) and, yields

$$\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P^i_{kh} = a_{lmn} P^i_{kh} + b_{lmn} \left(\delta^i_h \, y_k - \delta^i_k \, y_h \right) . \quad (2.4)$$

Transvecting equations (2.3) and (2.4) by g_{is}, using (1.4a), (1.7g), (1.7h) and (1.11c), we get

$$\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P_{jskh} = a_{lmn} P_{jskh} + b_{lmn} \left(g_{hs}\, g_{jk} - g_{ks}\, g_{jh} \right) - 2c_{lm} \mathcal{B}_r \, y^r \left(g_{hs}\, C_{jkn} - g_{ks}\, C_{jhn} \right) - 2d_{ln}\, \mathcal{B}_r \, y^r \left(g_{hs}\, C_{jkm} - g_{ks}\, C_{jhm} \right) - 2\,\mu_l\, \mathcal{B}_n \mathcal{B}_r y^r \left(g_{hs}\, C_{jkm} - g_{ks}\, C_{jhm} \right) , \quad (2.5)$$

and

$$\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P_{ksh} = a_{lmn} P_{ksh} + b_{lmn} \left(g_{hs}\, y_k - g_{ks}\, y_h \right) . \quad (2.6)$$

If and only if $\mathcal{B}_k g_{ij} = 0$.

Theorem 2.1 In G$\mathcal{B}$R-TIR- RF$_n$, Berwald's covariant derivative of the third order for the associate curvature tensor P_{jskh} and the tensor P_{ksh} are given by equations (2.5) and (2.6), respectively.

Contracting the indices i and h in the equations (2.3) and (2.4), using (1.2d), (1.7b), (1.7d), (1.8), (1.11a) and (1.11c), yields

$$\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P_{jk} = a_{lmn} P_{jk} + (n-1)\, b_{lmn}\, g_{jk} - 2c_{lm} \mathcal{B}_r \, y^r (n-1)\, C_{jkn} - 2d_{ln}\, \mathcal{B}_r \, y^r (n-1)\, C_{jkm} - 2\,\mu_l\, \mathcal{B}_n \mathcal{B}_r \, y^r (n-1)\, C_{jkm} , \quad (2.7)$$

and

$$\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P_k = a_{lmn} P_k + b_{lmn}\, (n-1) y_k . \quad (2.8)$$

The equations (2.7) and (2.8) show that the Ricci tensor P_{jk} and the curvature vector P_k can't vanish, because the vanishing for any one of them would imply the vanishing of the covariant tensor field b_{lmn} , i.e. $b_{lmn} = 0$, a contradiction.

Theorem 2.2 In G$\mathcal{B}$R-TIR- RF$_n$, the Ricci tensor P_{jk} and the curvature vector P_k are non-vanishing.

Transvecting the condition (2.8) by y^k, using (1.1b), (1.4a), and (1.7e), yields

$$\mathcal{B}_n \mathcal{B}_m \mathcal{B}_l P = a_{lmn} P + (n-1) b_{lmn} F^2 . \quad (2.9)$$

The equation (2.9) show that the scalar curvature P can't vanish, because the vanishing of it would imply the vanishing of the covariant tensor field b_{lmn} , i.e. $b_{lmn} = 0$, a contradiction.

Theorem 2.3 In G$\mathcal{B}$R-TIR- RF$_n$, the scalar curvature P is non-vanishing.

3. THE NECESSARY AND SUFFICIENT CONDITIONS

In this section, the necessary and sufficient conditions are obtained for some tensors to be generalized recurrent in $G\mathcal{B}$R-TIR- $\mathrm{RF_n}$.

Taking Berwald's covariant derivative of third order for equation (1.12) with respect to x^l , x^m and x^n, successively , using the conditions (2.3) and (1.12), yields

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(-S^i_{jkh|r}y^r\right) = -a_{lmn}\left(S^i_{jkh|r}y^r\right)$$
$$+\,b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$
$$-\,2c_{lm}\mathcal{B}_r y^r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})$$
$$-\,2d_{ln}\,\mathcal{B}_r\, y^r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})$$
$$-\,2\,\mu_l\,\mathcal{B}_n\mathcal{B}_r\, y^r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\,. \qquad (3.1)$$

Theorem 3.1 In $G\mathcal{B}$R-TIR- $\mathrm{RF_n}$, Berwald's covariant derivative of third order for the tensor $\left(S^i_{jkh|r}y^r\right)$ given in equation (3.1).

It is known that Cartan's third curvature tensor R^i_{jkh} and Cartan's second curvature tensor P^i_{jkh} are connected by the formula [3]:

$$P^i_{jkh} = R^i_{jkh} - \tfrac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right). \qquad (3.2)$$

Taking Berwald's covariant derivative of third order for equation (3.2) with respect to x^l , x^m and x^n, successively , yields

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l P^i_{jkh} = \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh}$$
$$-\tfrac{1}{3}\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(\delta^i_h R_{jk} - \delta^i_k R_{jh})\,. \qquad (3.3)$$

Using the condition (2.3) in equation (3.3), yields

$$a_{lmn}P^i_{jkh} + b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$
$$-\,2c_{lm}\mathcal{B}_r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\,y^r$$
$$-\,2\,d_{ln}\,\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\,y^r$$
$$-\,2\,\mu_l\,\mathcal{B}_n\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\,y^r =$$
$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} + \tfrac{1}{3}\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(\delta^i_h R_{jk} - \delta^i_k R_{jh}).$$

By using equation (3.2), the above equation implies to

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} + \tfrac{1}{3}\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(\delta^i_h R_{jk} - \delta^i_k R_{jh})$$
$$= a_{lmn}R^i_{jkh} - \tfrac{1}{3}\,a_{lmn}(\delta^i_h R_{jk} - \delta^i_k R_{jh})$$
$$+\,b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$
$$-\,2c_{lm}\mathcal{B}_r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})\,y^r$$
$$-\,2d_{ln}\,\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\,y^r$$
$$-\,2\,\mu_l\,\mathcal{B}_n\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\,y^r. \qquad (3.4)$$

This shows that

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} = a_{lmn}R^i_{jkh}$$
$$+\,b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$
$$-\,2c_{lm}\mathcal{B}_r y^r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})$$
$$-\,2d_{ln}\,\mathcal{B}_r y^r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})$$
$$-\,2\,\mu_l\,\mathcal{B}_n\mathcal{B}_r y^r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})\,. \qquad (3.5)$$

If and only if

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(\delta^i_h R_{jk} - \delta^i_k R_{jh})$$
$$= -\,a_{lmn}(\delta^i_h R_{jk} - \delta^i_k R_{jh})\,.$$

Therefore, using the above assumptions and mathematical analysis the following theorems have been derived.

Theorem 3.2 In $G\mathcal{B}$R-TIR- $\mathrm{RF_n}$, Cartan's third curvature tensor R^i_{jkh} is generalized trirecurrent if and only if the tensor $(\delta^i_h R_{jk} - \delta^i_k R_{jh})$ is trirecurrent.

Contracting the indices i and h in equation (3.5), using (1.2d), (1.8), (1.9d) and (1.11c), yields

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, R_{jk} = a_{lmn}R_{jk} + (n-1)b_{lmn}\, g_{jk} - 2c_{lm}\mathcal{B}_r\, y^r(n-1)\, C_{jkn} - 2d_{ln}\,\mathcal{B}_r\, y^r(n-1)\, C_{jkm} - 2\,\mu_l\,\mathcal{B}_n\mathcal{B}_r\, y^r(n-1)\, C_{jkm}\,. \tag{3.6}$$

Theorem 3.3 In $G\mathcal{B}$R-TIR- $\mathrm{RF_n}$, Berwald's covariant derivative of the third order for the Ricci tensor R_{jk} is given by equation (3.6).

Transvecting equation (3.5) by y^j,using (1.1a), (1.2c), (1.4a) and (1.9a), yields

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^i_{kh} = a_{lmn}H^i_{kh} + b_{lmn}\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right). \tag{3.7}$$

Thus, it leads to the following.

Theorem 3.4 In $G\mathcal{B}$R-TIR- $\mathrm{RF_n}$, Berwald's covariant derivative of the third order for the torsion tensor H^i_{kh} is given by (3.7).

Transvecting equation (3.7) by y^k,using (1.1b), (1.4a),(1.9f) and (1.11b), yields

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l H^i_h = a_{lmn}H^i_h + b_{lmn}\left(\delta^i_h\, F^2 - y^i y_h\right). \tag{3.8}$$

Thus, it leads to the following.

Theorem 3.5 In $G\mathcal{B}$R-TIR- $\mathrm{RF_n}$, Berwald's covariant derivative of the third order for the tensor H^i_h is given by equation (3.8).

Transvecting condition (3.6) by y^j, using (1.1a), (1.2c), (1.4a) and (1.9c), yields

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R_k = a_{lmn}\, R_k + (n-1)\, b_{lmn}\, y_k\,. \tag{3.9}$$

Theorem 3.6. In $G\mathcal{B}$R-TIR- $\mathrm{RF_n}$, Berwald's covariant derivative of the third order for the curvature vector R_k is given by equation (3.9) is non-vanishing if and only if the curvature vector R_k is also trirecurrent.

Using (1.13) and (1.14) in (3.2), we get

$$P^i_{jkh} = K^i_{jkh} + C^i_{js}H^s_{kh} - \tfrac{1}{3}\left\{\left(K_{jk} + C^r_{js}H^s_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{js}H^s_{hr}\right)\delta^i_k\right\}. \tag{3.10}$$

Taking Berwald's covariant derivative of third order for (3.10) with respect to x^l , x^m and x^n, successively , using condition (2.3) and (3.2), we get

$$\begin{aligned}\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K^i_{jkh} &= a_{lmn}K^i_{jkh} + b_{lmn}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - 2c_{lm}\mathcal{B}_r y^r\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2d_{ln}\,\mathcal{B}_r y^r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) \\ &\quad - 2\,\mu_l\,\mathcal{B}_n\mathcal{B}_r\, y^r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) + a_{lmn}\left(C^i_{js}H^s_{kh}\right) - \tfrac{1}{3}\, a_{lmn}\left\{\left(K_{jk} + C^r_{js}H^s_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{js}H^s_{hr}\right)\delta^i_k\right\} \\ &\quad - \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{js}H^s_{kh}\right) + \tfrac{1}{3}\,\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left\{\left(K_{jk} + C^r_{js}H^s_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{js}H^s_{hr}\right)\delta^i_k\right\}.\end{aligned} \tag{3.11}$$

This shows that

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K^i_{jkh} = a_{lmn}\, K^i_{jkh} + b_{lmn}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - 2c_{lm}\mathcal{B}_r y^r\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - 2d_{ln}\,\mathcal{B}_r y^r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) - 2\,\mu_l\,\mathcal{B}_n\mathcal{B}_r y^r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right). \tag{3.12}$$

If and only if

$$\tfrac{1}{3}\,\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left\{\left(K_{jk} + C^r_{js}H^s_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{js}H^s_{hr}\right)\delta^i_k\right\} - \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^i_{js}H^s_{kh}\right) = \tfrac{1}{3}a_{lmn}\left\{\left(K_{jk} + C^r_{js}H^s_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{js}H^s_{hr}\right)\delta^i_k\right\} - a_{lmn}\left(C^i_{js}H^s_{kh}\right). \tag{3.13}$$

Thus, we conclude

Theorem 3.7. In $G\mathcal{B}$R-TIR-$\mathrm{RF_n}$, Cartan's fourth curvature tensor K^i_{jkh} is generalized trirecurrent if and only if equation (3.13) holds.

Contracting the indices i and h in the condition (3.11), using (1.2d), (1.8), (1.10a) and (1.11c), we get

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K_{jk} = a_{lmn}K_{jk} + (n-1)\, b_{lmn}\, g_{jk} - 2c_{lm}\mathcal{B}_r y^r(n-1)C_{jkn} - 2d_{ln}\,\mathcal{B}_r y^r(n-1)C_{jkm} -2\,\mu_l\,\mathcal{B}_n\mathcal{B}_r y^r(n-1)C_{jkm} + a_{lmn}\left(C^t_{js}H^s_{kt}\right) - \tfrac{1}{3}\, a_{lmn}\{\, n\,\left(K_{jk} + C^r_{js}H^s_{kr}\right) - \left(K_{jh} + C^r_{js}H^s_{tr}\right)\delta^t_k\,\} - \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^t_{js}H^s_{kt}\right) + \tfrac{1}{3}\,\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\{n\left(K_{jk} + C^r_{js}H^s_{kr}\right) - \left(K_{jh} + C^r_{js}H^s_{tr}\right)\delta^t_k\,\}\ . \tag{3.14}$$

This shows that

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,K_{jk} = a_{lmn}\,K_{jk} + (n-1)\,b_{lmn}\,g_{jk} - 2\,(n-1)\,c_{lm}\mathcal{B}_r C_{jkn}\,y^r - 2(n-1)\,d_{ln}\mathcal{B}_r C_{jkm}\,y^r -2(n-1)\,\mu_l\,\mathcal{B}_n\mathcal{B}_r C_{jkm}\,y^r\ . \tag{3.15}$$

If and only if

$$\tfrac{1}{3}\,\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\{n\left(K_{jk} + C^r_{js}H^s_{kr}\right) - \left(K_{jh} + C^r_{js}H^s_{tr}\right)\delta^t_k\,\} - \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^t_{js}H^s_{kt}\right) = \tfrac{1}{3}\,a_{lmn}\{n\left(K_{jk} + C^r_{js}H^s_{kr}\right) - \left(K_{jh} + C^r_{js}H^s_{tr}\right)\delta^t_k\,\} - a_{lmn}\left(C^t_{js}H^s_{kt}\right)\ . \tag{3.16}$$

Thus, we conclude

Theorem 3.8. In $G\mathcal{B}$R-TIR-RF$_n$, Berwald's covariant derivative of the third order for the Ricci tensor K_{jk} (for Cartan's fourth curvature tensor K^i_{jkh}) is given by (3.15) if and only if the equation (3.16) holds.

Transvecting (3.14) by y^k, using (1.1a), (1.2c), (1.4a), (1.9f), (1.10c), and (1.11b), we get

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K_j = a_{lmn}K_j + (n-1)\,b_{lmn}\,y_j + a_{lmn}\left(C^t_{js}H^s_t\right) - \tfrac{1}{3}\,a_{lmn}\{n\left(K_j + C^r_{js}H^s_r\right) -\left(K_{jh} + C^r_{js}H^s_{tr}\right)y^t\} - \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^t_{js}H^s_t\right) + \tfrac{1}{3}\,\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\{n\left(K_j + C^r_{js}H^s_r\right) - \left(K_{jh} + C^r_{js}H^s_{tr}\right)y^t\}\,. \tag{3.17}$$

This shows that

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l K_j = a_{lmn}K_j + (n-1)b_{lmn}y_j\,. \tag{3.18}$$

If and only if

$$\tfrac{1}{3}\,\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\{n\left(K_j + C^r_{js}H^s_r\right) - \left(K_{jh} + C^r_{js}H^s_{tr}\right)y^t\} - \mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(C^t_{js}H^s_t\right) = \tfrac{1}{3}\,a_{lmn}\{n\left(K_j + C^r_{js}H^s_r\right) - \left(K_{jh} + C^r_{js}H^s_{tr}\right)y^t\} - a_{lmn}\left(C^t_{js}H^s_t\right)\ . \tag{3.19}$$

The equation (3.18), shows that the curvature vector K_j can't vanish, because the vanishing of it would imply the vanishing of the covariant vector field b_{lmn}, i.e. $b_{lmn} = 0$, a contradiction.

Thus, we conclude

Theorem 3.9. In $G\mathcal{B}$R-TIR-RF$_n$,the curvature vector K_j (for Cartan's fourth curvature tensor K^i_{jkh}) is non-vanishing if and only if the equation (3.19) holds.

4. DISCUSSIONS OF RESULTS

In discussion of the results of section 2, the $G\mathcal{B}$R-TIR-RF$_n$ is introduced.

It is found by Berwald's covariant derivative of the third order for Cartan tensors and is shown in the non-vanishing of the same tensors. Also in discussion of the results of section 3, the necessary and sufficient conditions are obtained for some curvature tensors to be generalized trirecurrent and of Berwald's covariant derivative of the third order for Cartan tensors are obtained and also the condition for non-vanishing of the same tensors.

More results is important for work in Finsler space.

5. CONCLUSIONS AND RECOMMENDAITIONS

A Finsler space is called generalized $\mathcal{B}$P-trirecurrent if it satisfies the condition (2.3).

In G$\mathcal{B}$R-TIR-RF$_n$, Berwald's covariant derivative of the third order for the torsion tensor H^i_{kh} , the deviation tensor H^i_h and the Ricci tensor P_{jk} are given by the equations (3.7), (3.8) and (2.7), respectively. In G$\mathcal{B}$R-TIR-RF$_n$, the curvature vector P_k , the scalar curvature P and the curvature vector R_j are non-vanishing. In G$\mathcal{B}$R-TIR-RF$_n$, the necessary and sufficient condition of Cartan's thirth curvature tensor R^i_{jkh} to be generalized trirecurrent is given by the equation (3.5). In G$\mathcal{B}$R-TIR-RF$_n$, the necessary and sufficient conditions of Berwald's covariant derivative of the third order for the Ricci tensor K_{jk} , is given by equation (3.15).

In G$\mathcal{B}$R-TIR-RF$_n$, the necessary and sufficient conditions of non-vanishing for the curvature vector K_j is given by equation (3.18). In G$\mathcal{B}$R-TIR-RF$_n$, the necessary and sufficient condition of Cartan's fourth curvature tensor K^i_{jkh} to be generalized trirecurrent is given by the equation (3.12).

In G$\mathcal{B}$R-TIR-RF$_n$, the necessary and sufficient conditions of Berwald's covariant derivative of the third order for the Ricci tensor K_{jk} .

Authors recommend the need for continuing research and development in generalized $\mathcal{B}$R-trirecurrent spaces and interlard it with the properties of special spaces for Finsler space.

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حول تعميمات المؤتر $\mathcal{B}P$ احادي المعاودة في فضاء فنسلر من الرتبة الثالثة

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الملخص: في هذه الورقة، قدمنا تعريف للتعميم الثلاثي لفضاء فنسلر F_n الذي يحقق المونر التقوسي الثاني لكارتان P^i_{jkh} باستخدام مشتقة برو الد ثلاثية المعاودة وأطلقنا على هذا الفضاء بتعميم فضاء فنسلر $\mathcal{B}P$ - ثلاثي المعاودة ورمزنا إليه بالرمز $G\mathcal{B}\text{P-TIR-RF}_\text{n}$. كذلك أوجدنا الشرط اللازم والكافي لبعض الموترات بمفهوم برو الد لكي تكون معممة ثلاثية المعاودة في هذا الفضاء.

الكلمات المفتاحية: المؤتر الثاني لكارتان - تعميم $\mathcal{B}P$ ثلاثي المعاودة - مشتقة بارولاد من الرتبة الثالثة