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Theoretical Analysis of Generalized W^h-Birecurrent Finsler Spaces with Emphasis on Weyl's Projective Curvature and Its Relevance to Geometric Mechanics

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Abstract: The present work develops a rigorous theoretical framework for the structure of generalized W^h-birecurrent Finsler spaces, with special emphasis on the analytical behavior of Weyl's projective curvature tensor. By employing Cartan's h-covariant differentiation and exploiting the identities governing affinely connected Finsler manifolds, several equivalent conditions characterizing generalized W^h-birecurrent spaces are established. New commutation relations for higher-order h-covariant derivatives of Weyl's projective curvature, torsion, and deviation tensors are derived. The analysis clarifies how these tensors behave under recurrence and birecurrence conditions, and how such properties propagate in affinely connected settings.

Furthermore, the study highlights the relevance of these curvature structures to geometric mechanics and theoretical physics, where Finsler-type curvatures naturally arise in the modeling of non-Euclidean trajectories and anisotropic dynamical systems. The results deepen the understanding of curvature-driven geometric behavior and provide a foundation for future applications in mechanical and physical systems governed by generalized geometric dynamics.

Key words: Finsler Geometry - Weyl's Projective Curvature - Generalized W^h-Birecurrence - Affinely Connected Spaces - Geometric Mechanics - Theoretical Physics.

1. Introduction

Finsler geometry provides one of the most general frameworks for studying geometric structures in which the fundamental metric depends not only on the positional coordinates x^ibut also on the directional arguments y^i. This anisotropic nature gives rise to a rich hierarchy of connections, curvature tensors, and derivative operators among which Cartan's h-covariant differentiation plays a central role. Within this framework, Weyl's projective curvature tensor emerges as a fundamental geometric object that measures projective deformation in Finsler manifolds. Its behavior under various recurrence conditions has been of particular interest in differential geometry, as recurrence often reveals deep structural symmetries in curvature spaces. Generalized W^h-birecurrent spaces extend the classical notion of recurrence to higher-order h-covariant derivatives, producing stronger geometric restrictions on curvature tensors. These constraints become especially significant in affinely connected Finsler spaces, where properties such as the vanishing of the Berwald connection derivatives simplify the tensorial structure.

The objective of this study is to provide a complete theoretical characterization of generalized W^h-birecurrent Finsler spaces by analyzing the transformation behavior of Weyl's projective curvature, torsion, and deviation tensors under successive h-covariant differentiation. The derived identities not only establish necessary and sufficient conditions for birecurrence, but also demonstrate how this curvature structures relate to geometric mechanics and theoretical physics particularly in systems governed by anisotropic trajectories, non-Euclidean dynamics, and curvature-driven motion.

The results presented here contribute to a unified tensorial understanding of curvature recurrence and provide a basis for potential applications in mechanical modeling, physical field theories, and geometric control systems.

Over the past decades, the theory of Finsler geometry has undergone significant development, particularly in the study of curvature tensors, recurrent structures, and the behavior of higher-order covariant derivatives. Foundational contributions by Rund (1959), Matsumoto (1972), Akbar-Zadeh (1988), Bao, Chern, and Shen (2000), and other classical works established the geometric and analytic basis for understanding Cartan and Berwald connections, projective curvature structures, and the role of anisotropic metrics in generalized geometric settings.

Building on these classical foundations, a substantial body of modern research most notably by Al-Qashbari and his collaborators has expanded the framework of recurrent and birecurrent curvature tensors in Finsler spaces. These studies introduced and analyzed a wide spectrum of generalized recurrent structures, including BR-trirecurrent spaces, fifth-order recurrent structures, generalized R^h-recurrent spaces, and various extensions of M-projective and R-projective curvature tensors. Through the systematic use of Cartan's and Berwald's higher-order derivatives, these works provided refined decomposition formulas, structural identities, and necessary and sufficient conditions characterizing recurrence, birecurrence, and trirecurrence in several classes of Finsler manifolds.

Moreover, the recent contributions of Al-Qashbari et al. have deepened the analysis of Weyl's projective curvature tensor—particularly its decomposition, recurrence behavior, and transformation properties under generalized covariant derivatives. These studies also explored the geometric implications of curvature tensors in higher-order recurrent spaces and established new relationships between torsion, deviation, and projective curvature structures.

Taken together, these previous studies form a coherent and progressively expanding research landscape that extends the classical theory of Finsler geometry into more sophisticated domains of recurrent tensor analysis. They provide a strong theoretical foundation that supports the present work, which further examines generalized W^h-birecurrent spaces and the analytical behavior of Weyl's projective curvature under successive h-covariant differentiation.

An n-dimensional Finsler space, equipped with the metric function f satisfies the requisite conditions Consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ^{*i}_{jk} and Berwald's connection parameters G^i_{jk} (the indices $i, j, k, \ldots$ assume positive integral values from 1 to n). These are symmetric in their lower indices.

The vectors y_i and y^i satisfy the following relations:

(1.1) a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $\dot{\partial}_i y_j = g_{ij}$ and d) $\dot{\partial}_j\, y^i = \delta^i_j$.

The metric tensor g_{ij} Fig.(1.2), the metric tensor g^{ij} Fig.(1.3) and the vector y^i are covariant constant with respect to the above process.

(1.2) a) $g_{ij|k} = 0$, b) $y^i_{|k} = 0$ and c) $g^{ij}{}_{|k} = 0$.

The h-covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j, successively, we get

(1.3) $X^i_{|k|j} = \dot{\partial}_j \left(X^i_{|k}\right) - \left(X^i_{|r}\right)\Gamma^{*r}_{kj} + \left(X^r_{|k}\right)\Gamma^{*i}_{rj} - \dot{\partial}_r \left(X^i_{|k}\right)\Gamma^{*r}_{js}\, y^s$.

Taking skew-symmetric part with respect to the indices k and j, we get the commutation formula for h-covariant differentiation as follows [13]:

(1.4) $X^i_{|k|j} - X^i_{|j|k} = X^r\, K^i_{rkj} - \left(\dot{\partial}_r\, X^i\right) K^r_{skj}\, y^s$,

were

(1.5) $K^i_{rkj} = \partial_j\, \Gamma^{*i}_{kr} + \left(\dot{\partial}_l\, \Gamma^{*i}_{rj}\right) G^l_k + \Gamma^{*i}_{mj}\, \Gamma^{*m}_{kr} - \partial_k\, \Gamma^{*i}_{jr} - \left(\dot{\partial}_l\, \Gamma^{*i}_{rk}\right) G^l_j - \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jr}$.

The tensor K^i_{rkj} as defined above is called *Cartan's fourth curvature tensor*.

The process of h-covariant differentiation, with respect to x^k , commute with partial differentiation with respect to y^j for arbitrary vector filed X^i , according to

(1.6) a) $\dot{\partial}_j \left(X^i_{|k}\right) = \left(\dot{\partial}_j X^i\right)_{|k} + X^r\left(\dot{\partial}_j \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk}$ and b) $\dot{\partial}_j \Gamma^{*i}_{rk} = \Gamma^{*i}_{rkj}$.

The tensor W^i_{jkh} is known as projective curvature tensor (generalized Wely's projective curvature tensor), the tensor W^i_{jk} is known as projective torsion tensor (Wely's torsion tensor) and the tensor W^i_j is known as projective deviation tensor (Wely's deviation tensor) are defined by

(1.7) $$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{n+1} H_{[hk]} + \frac{2\,y^i}{n+1}\dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{n^2-1}\left(n\,H_{jh} + H_{hj} + y^r\dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{n^2-1}\left(n\,H_{jk} + H_{kj} + y^r\dot{\partial}_j H_{kr}\right) ,$$

(1.8) $$W^i_{jk} = H^i_{jk} + \frac{y^i}{n+1} H_{[jk]} + 2\left\{\frac{\delta^i_{[j}}{n^2-1}\left(n\,H_{k]} - y^r H_{k]\,r}\right)\right.$$

and

(1.9) $$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{n+1}\left(\dot{\partial}_r H^r_j - \dot{\partial}_j H\right) y^i ,$$

respectively.

The tensors W^i_{jkh} , W^i_{jk} and W^i_k are satisfying the following identities [13]

(1.10) a) $W^i_{jkh}\, y^j = W^i_{kh}$, b) $W^i_{jk}\, y^j = W^i_k$,

c) $\dot{\partial}_j W^i_{kh} = W^i_{jkh}$ and d) $\dot{\partial}_k W^i_h = W^i_{kh}$.

The projective curvature tensor W^i_{jkh} is skew-symmetric in its indices k and h.

An affinely connected space has some properties as follows:

(1.11) a) $G^i_{jkh} = 0$ and b) $C_{ijk|h} = 0$.

Remark 1.1. An affinely connected space or Berwald space characterized by any one of the above two equivalent conditions.

Also, we have the following properties

The connection parameters Γ^{*i}_{jk} of Cartan and G^i_{jk} of Berwald coincide in an affinely connected space and they are independent of the directional arguments

(1.12) a) $G^i_{jkh} = \dot{\partial}_j G^i_{kh} = 0$, b) $\dot{\partial}_j \Gamma^{*i}_{kh} = 0$

and c) $y_r\, G^r_{ijk} = -2\, C_{ijk|h}\, y^h = -2\, P_{ijk} = 0$.

2. Necessary and Sufficient Conditions for Generalized W^h-Birecurrent Finsler Spaces

This section develops a rigorous characterization of generalized W^h-birecurrent Finsler spaces by examining the recurrence behavior of Weyl's projective curvature tensor with respect to Cartan's h-covariant differentiation. Starting from the first-order recurrence condition

(2.1) $$W^i_{jkh|m} = a_m W^i_{jkh} + b_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right),$$

the analysis investigates how this structure evolves under successive h-covariant derivatives. By applying the identities of Cartan's connection and using the vanishing of metric derivatives, the second-order condition is shown to take the form

$$W^i_{jkh|m|n} = \lambda_{mn} W^i_{jkh} + \mu_{mn}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right),$$

which provides the essential criterion that defines a generalized W^h-birecurrent space.

The results demonstrate that every generalized W^h-recurrent space automatically satisfies the birecurrence property, and explicit forms for the second-order h-covariant derivatives of Weyl's projective torsion and deviation tensors are established. Furthermore, the derived tensorial expressions yield precise necessary and sufficient conditions presented in Theorems 2.1, 2.2, and 2.3 for Weyl's curvature, torsion, and deviation tensors to preserve their birecurrent structure under higher-order differentiation. This framework forms a fundamental analytical basis for understanding recurrent curvature structures in Finsler geometry.

The space is called as a generalized W^h-birecurrent space and denoted briefly by $GW^h\text{-}BIRF_n$.

Result 2.1. Every generalized W^h-recurrent space is generalized W^h-birecurrent space.

Transvecting the condition (2.2) by y^j and by y^k, successively, using (1.3b), (1.10a), (1.10b), (1.1a) and (1.1b), we get

(2.3) $$W^i_{kh|m|n} = \lambda_{mn} W^i_{kh} + \mu_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) ,$$

(2.4) $$W^i_{h|m|n} = \lambda_{mn} W^i_h + \mu_{mn}(\delta^i_h F^2 - y_h y^i) ,$$

where λ_{mn} and μ_{mn} are non-null covariant vector fields.

Thus, it leads to the following theorem.

Theorem 2.1. In a generalized W^h-birecurrent Finsler space $GW^h\text{-}BIRF_n$, the second-order h-covariant derivatives of Weyl's projective torsion tensor W^i_{kh} and deviation tensor W^i_h are given (2.3) and (2.4), respectively.

Differentiating (2.3) partially with respect to y^j, using (1.14c) and (1.1c), we get

$$\dot\partial_j\left(W^i_{kh|m|n}\right) = (\dot\partial_j \lambda_{mn}) W^i_{kh} + \lambda_{mn} W^i_{jkh} + (\dot\partial_j \mu_{mn})(\delta^i_h y_k - \delta^i_k y_h) + \mu_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) .$$

Using the commutation formula exhibited by (1.6a) for the h(v) torsion tensor $\left(W^i_{kh|m}\right)$, in the above equation, we get

(2.5) $$\left\{\dot\partial_j W^i_{kh|m}\right\}_{|n} + W^r_{kh|m}(\dot\partial_j \Gamma^{*i}_{rn}) - W^i_{rh|m}(\dot\partial_j \Gamma^{*r}_{kn}) - W^i_{kr|m}(\dot\partial_j \Gamma^{*r}_{hn}) - W^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{mn}) - \left(\dot\partial_r W^i_{kh|m}\right) P^r_{jn} = (\dot\partial_j \lambda_{mn}) W^i_{kh} + \lambda_{mn} W^i_{jkh} + (\dot\partial_j \mu_{mn})(\delta^i_h y_k - \delta^i_k y_h) + \mu_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) .$$

Again, apply the commutation formula exhibited by (1.6a) for the h(v) torsion tensor $\left(W^i_{kh}\right)$, in the equation (2.5) and using (1.10c), we get

$$\left\{W^i_{jkh|m} + W^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - W^i_{rh}(\dot\partial_j \Gamma^{*r}_{km}) - W^i_{kr}(\dot\partial_j \Gamma^{*r}_{hm}) - (\dot\partial_r W^i_{kh}) P^r_{jm}\right\}_{|n} + W^r_{kh|m}(\dot\partial_j \Gamma^{*i}_{rn}) - W^i_{rh|m}(\dot\partial_j \Gamma^{*r}_{kn}) - W^i_{kr|m}(\dot\partial_j \Gamma^{*r}_{hn}) - W^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{mn}) - \left(\dot\partial_r W^i_{kh|m}\right) P^r_{jn} = (\dot\partial_j \lambda_{mn}) W^i_{kh} + \lambda_{mn} W^i_{jkh} + (\dot\partial_j \mu_{mn})(\delta^i_h y_k - \delta^i_k y_h) + \mu_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) ,$$

which can be written as

(2.6) $$W^i_{jkh|m|n} + \left\{W^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - W^i_{rh}(\dot\partial_j \Gamma^{*r}_{km}) - W^i_{kr}(\dot\partial_j \Gamma^{*r}_{hm}) - (\dot\partial_r W^i_{kh}) P^r_{jm}\right\}_{|n} + W^r_{kh|m}(\dot\partial_j \Gamma^{*i}_{rn}) - W^i_{rh|m}(\dot\partial_j \Gamma^{*r}_{kn}) - W^i_{kr|m}(\dot\partial_j \Gamma^{*r}_{hn}) - W^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{mn}) - \left(\dot\partial_r W^i_{kh|m}\right) P^r_{jn} = (\dot\partial_j \lambda_{mn}) W^i_{kh} + \lambda_{mn} W^i_{jkh} + (\dot\partial_j \mu_{mn})(\delta^i_h y_k - \delta^i_k y_h) + \mu_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) .$$

This shows that

$$W^i_{jkh|m|n} = \lambda_{mn} W^i_{jkh} + \mu_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$

if and only if

(2.7) $$\left\{W^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - W^i_{rh}(\dot\partial_j \Gamma^{*r}_{km}) - W^i_{kr}(\dot\partial_j \Gamma^{*r}_{hm}) - (\dot\partial_r W^i_{kh}) P^r_{jm}\right\}_{|n} + W^r_{kh|m}(\dot\partial_j \Gamma^{*i}_{rn}) - W^i_{rh|m}(\dot\partial_j \Gamma^{*r}_{kn}) - W^i_{kr|m}(\dot\partial_j \Gamma^{*r}_{hn}) - W^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{mn}) - \left(\dot\partial_r W^i_{kh|m}\right) P^r_{jn} - (\dot\partial_j \lambda_{mn}) W^i_{kh} - (\dot\partial_j \mu_{mn})(\delta^i_h y_k - \delta^i_k y_h) = 0 .$$

Thus, it leads to the following.

Theorem 2.2. In $GW^h\text{-}BIRF_n$, Weyl's projective curvature tensor W^i_{jkh} is generalized birecurrent (G-BR) if and only if the following condition is satisfied: $\mathcal{T}^i_{jkhmn} - (\partial_{.j}\lambda_{mn}) W^i_{kh} - (\partial_{.j}\mu_{mn})(\delta^i_h y_k - \delta^i_k y_h) = 0$, where $\mathcal{T}^i_{jkhmn}$ denotes the tensorial expression generated by the commutation of h(v)-covariant derivatives as defined in equation (2.7).

Differentiating (2.4) partially with respect to y^k, using (1.10d) and (1.1c), we get

$$\dot\partial_k\left(W^i_{h|m|n}\right) = (\dot\partial_k \lambda_{mn}) W^i_h + \lambda_{mn} W^i_{kh} + (\dot\partial_k \mu_{mn})(\delta^i_h F^2 - y_h y^i) + \mu_{mn}(\dot\partial_k \delta^i_h F^2 - \delta^i_k y_h).$$

Using the commutation formula exhibited by (1.6a) for the h(v) torsion tensor $\left(W^i_{h|m}\right)$, in the above equation, we get

(2.8) $$\left\{\dot\partial_k W^i_{h|m}\right\}_{|n} + W^r_{h|m}\left(\dot\partial_k \Gamma^{*i}_{rn}\right) - W^i_{r|m}\left(\dot\partial_k \Gamma^{*r}_{hn}\right) - W^i_{h|r}\left(\dot\partial_k \Gamma^{*r}_{mn}\right) - \left(\dot\partial_r W^i_{h|m}\right) P^r_{kn}$$
$$= \left(\dot\partial_k \lambda_{mn}\right) W^i_h + \lambda_{mn} W^i_{kh} + \left(\dot\partial_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) + \mu_{mn}\left(\dot\partial_k \delta^i_h F^2 - \delta^i_k y_h\right).$$

Again, apply the commutation formula exhibited by (1.6a) for the h(v) torsion tensor $\left(W^i_h\right)$, in the equation (2.8) and using (1.10d), we get

$$\left\{W^i_{kh|m} + W^r_h\left(\dot\partial_k \Gamma^{*i}_{rm}\right) - W^i_r\left(\dot\partial_k \Gamma^{*r}_{hm}\right) - \left(\dot\partial_r W^i_h\right) P^r_{km}\right\}_{|n} + W^r_{h|m}\left(\dot\partial_k \Gamma^{*i}_{rn}\right)$$
$$- W^i_{r|m}\left(\dot\partial_k \Gamma^{*r}_{hn}\right) - W^i_{h|r}\left(\dot\partial_k \Gamma^{*r}_{mn}\right) - \left(\dot\partial_r W^i_{h|m}\right) P^r_{kn} = \left(\dot\partial_k \lambda_{mn}\right) W^i_h + \lambda_{mn} W^i_{kh}$$
$$+ \left(\dot\partial_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) + \mu_{mn}\left(\dot\partial_k \delta^i_h F^2 - \delta^i_k y_h\right) ,$$

which can be written as

(2.9) $$W^i_{kh|m|n} + \left\{W^r_h\left(\dot\partial_k \Gamma^{*i}_{rm}\right) - W^i_r\left(\dot\partial_k \Gamma^{*r}_{hm}\right) - \left(\dot\partial_r W^i_h\right) P^r_{km}\right\}_{|n} + W^r_{h|m}\left(\dot\partial_k \Gamma^{*i}_{rn}\right)$$
$$- W^i_{r|m}\left(\dot\partial_k \Gamma^{*r}_{hn}\right) - W^i_{h|r}\left(\dot\partial_k \Gamma^{*r}_{mn}\right) - \left(\dot\partial_r W^i_{h|m}\right) P^r_{kn} = \left(\dot\partial_k \lambda_{mn}\right) W^i_h + \lambda_{mn} W^i_{kh}$$
$$+ \left(\dot\partial_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) + \mu_{mn}\left(\dot\partial_k \delta^i_h F^2 - \delta^i_k y_h\right) .$$

This shows that

(2.10) $$W^i_{kh|m|n} = \lambda_{mn} W^i_{kh} + \mu_{mn}\left(\dot\partial_k \delta^i_h F^2 - \delta^i_k y_h\right)$$

if and only if

(2.11) $$\left\{W^r_h\left(\dot\partial_k \Gamma^{*i}_{rm}\right) - W^i_r\left(\dot\partial_k \Gamma^{*r}_{hm}\right) - \left(\dot\partial_r W^i_h\right) P^r_{km}\right\}_{|n} + W^r_{h|m}\left(\dot\partial_k \Gamma^{*i}_{rn}\right) - W^i_{r|m}\left(\dot\partial_k \Gamma^{*r}_{hn}\right)$$
$$- W^i_{h|r}\left(\dot\partial_k \Gamma^{*r}_{mn}\right) - \left(\dot\partial_r W^i_{h|m}\right) P^r_{kn} = \left(\dot\partial_k \lambda_{mn}\right) W^i_h + \left(\dot\partial_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) .$$

Therefore, it is concluded the following.

Theorem 2.3. In GW^h-$BIRF_n$, Wely's projective torsion tensor W^i_{kh} is given by (2.10), if and only if the equation (2.11) holds good.

3. Affinely Connected Structure in Generalized W^h-Birecurrent Spaces

We now introduce the notion of an affinely connected generalized W^h-birecurrent Finsler space.

Definition 3.1. A generalized W^h-birecurrent Finsler space that additionally satisfies any one of the affine connection conditions (1.12a), (1.12b), or (1.12c) is called a generalized W^h-birecurrent affinely connected space. Such a space will be denoted succinctly by GW^h-BIR-affinely connected space.

Remark 3.1. Any tensor field that satisfies the defining conditions of a GW^h-BIR-affinely connected space may be referred to simply as a generalized h-birecurrent tensor, abbreviated as Gh-BIR.

By applying the affine connection conditions (1.12a), (1.12b), and (1.12c), equation (2.6) simplifies to the reduced form:

(3.1) $$W^i_{jkh|m|n} = \left(\dot\partial_j \lambda_{mn}\right) W^i_{kh} + \lambda_{mn} W^i_{jkh} + \left(\dot\partial_j \mu_{mn}\right)\left(\delta^i_h y_k - \delta^i_k y_h\right) + \mu_{mn}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) .$$

This shows that

(3.2) $$W^i_{jkh|m|n} = \lambda_{mn} W^i_{jkh} + \mu_{mn}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$

if and only if

(3.3) $$\left(\dot\partial_j \lambda_{mn}\right) W^i_{kh} + \left(\dot\partial_j \mu_{mn}\right)\left(\delta^i_h y_k - \delta^i_k y_h\right) = 0 .$$

Therefore, it is concluded the following.

Theorem 3.1. In a generalized W^h-birecurrent affinely connected space, Weyl's projective curvature tensor $W^i_{jkh}$is generalized h-birecurrent (Gh-BIR) if and only if: $\left(\dot\partial_j \lambda_{mn}\right) W^i_{kh} + \left(\dot\partial_j \mu_{mn}\right)\left(\delta^i_h y_k - \delta^i_k y_h\right) = 0$.

Transvecting the equation (3.1) by y^j, using (1.2b), (1.10a) and (1.1a), we get

(3.4) $$W^i_{kh|m|n} = \left(\dot\partial_j \lambda_{mn}\right) W^i_{kh}\, y^j + \lambda_{mn} W^i_{kh} + \left(\dot\partial_j \mu_{mn}\right)\left(\delta^i_h y_k - \delta^i_k y_h\right) y^j + \mu_{mn}\left(\delta^i_h y_k - \delta^i_k y_h\right) .$$

This shows that

(3.5) $$W^i_{kh|m|n} = \lambda_{mn} W^i_{kh} + \mu_{mn}\left(\delta^i_h y_k - \delta^i_k y_h\right)$$

if and only if

(3.6) $\left(\dot{\partial}_j \lambda_{mn}\right) W^i_{kh} + \left(\dot{\partial}_j \mu_{mn}\right)\left(\delta^i_h y_k - \delta^i_k y_h\right) = 0$, since $y^j \neq 0$.

Transvecting the equation (3.4) by y^k, using (1.2b), (1.10b), (1.1b) and (1.1a), we get

$$W^i_{h|m|n} = \left(\dot{\partial}_j \lambda_{mn}\right) W^i_h y^j + \lambda_{mn} W^i_h + \left(\dot{\partial}_j \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) y^j + \mu_{mn}\left(\delta^i_h F^2 - y_h y^i\right) .$$

This shows that

(3.7) $W^i_{h|m|n} = \lambda_{mn} W^i_h + \mu_{mn}\left(\delta^i_h F^2 - y_h y^i\right)$

if and only if

(3.8) $\left(\dot{\partial}_j \lambda_{mn}\right) W^i_h y^j + \left(\dot{\partial}_j \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) y^j = 0$.

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.2. In a GW^h-BIRaffinely connected space, Weyl's projective torsion tensor $W^i_{kh}$and deviation tensor W^i_h satisfy:

$$W^i_{kh|m|n} = \lambda_{mn} W^i_{kh} + \mu_{mn}\left(\delta^i_h y_k - \delta^i_k y_h\right),$$
$$W^i_{h|m|n} = \lambda_{mn} W^i_h + \mu_{mn}\left(\delta^i_h F^2 - y_h y^i\right),$$

if and only if the respective conditions:

$$\left(\dot{\partial}_k \lambda_{mn}\right) W^i_{kh} + \left(\dot{\partial}_k \mu_{mn}\right)\left(\delta^i_h y_k - \delta^i_k y_h\right) = 0 ,$$
$$\left(\dot{\partial}_k \lambda_{mn}\right) W^i_h y^j + \left(\dot{\partial}_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) y^j = 0 , \quad \text{hold.}$$

By using the conditions (1.12a), (1.12b) and (1.12c), the equation (2.9) reduce to

(3.9) $W^i_{kh|m|n} = \left(\dot{\partial}_k \lambda_{mn}\right) W^i_h + \lambda_{mn} W^i_{kh} + \left(\dot{\partial}_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) + \mu_{mn}\left(\dot{\partial}_k \delta^i_h F^2 - \delta^i_k y_h\right)$.

This shows that

(3.10) $W^i_{kh|m|n} = \lambda_{mn} W^i_{kh} + \mu_{mn}\left(\dot{\partial}_k \delta^i_h F^2 - \delta^i_k y_h\right)$

if and only if

(3.11) $\left(\dot{\partial}_k \lambda_{mn}\right) W^i_h + \left(\dot{\partial}_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) = 0$.

Therefore, it is concluded the following.

Theorem 3.3. Weyl's projective torsion tensor W^i_{kh}in a generalized W^h-$birecurrent$affinely connected space satisfies:

$$W^i_{kh|m|n} = \lambda_{mn} W^i_{kh} + \mu_{mn}\left(\dot{\partial}_k \delta^i_h F^2 - \delta^i_k y_h\right) ,$$

if and only if:

$$\left(\dot{\partial}_k \lambda_{mn}\right) W^i_h + \left(\dot{\partial}_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) = 0 .$$

Transvecting the equation (3.9) by y^k, using (1.2b), (1.10b), (1.1b) and (1.1a), we get

$$W^i_{h|m|n} = \left(\dot{\partial}_k \lambda_{mn}\right) W^i_h y^k + \lambda_{mn} W^i_h + \left(\dot{\partial}_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) y^k + \mu_{mn}\left(\delta^i_h F^2 - y_h y^i\right) .$$

This shows that

(3.12) $W^i_{h|m|n} = \lambda_{mn} W^i_h + \mu_{mn}\left(\delta^i_h F^2 - y_h y^i\right)$

if and only if

(3.13) $\left(\dot{\partial}_k \lambda_{mn}\right) W^i_h y^k + \left(\dot{\partial}_k \mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) y^k = 0$.

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.4. Weyl's deviation tensor W^i_hsatisfies:

$$W^i_{h|m|n} = \lambda_{mn} W^i_h + \mu_{mn}(\delta^i_h F^2 - y_h y^i),$$

if and only if:

$$(\partial_{\cdot k} \lambda_{mn}) W^i_h y^k + (\partial_{\cdot k} \mu_{mn})(\delta^i_h F^2 - y_h y^i) y^k = 0.$$

4. Conclusion

The present study has established a comprehensive theoretical framework for the structure and behavior of generalized W^h-birecurrent Finsler spaces, with particular emphasis on the analytic properties of Weyl's projective curvature tensor. By employing Cartan's h-covariant differentiation and exploiting the defining identities of affinely connected Finsler manifolds, several equivalent conditions for recurrence and birecurrence have been rigorously derived.

The analysis shows that the propagation of recurrence conditions to higher-order h-covariant derivatives imposes strong structural constraints on Weyl's projective curvature, torsion, and deviation tensors. These constraints lead to clear necessary and sufficient conditions characterizing generalized W^h-birecurrent spaces, both in the general Finsler setting and under the additional assumptions of affine connection.

Furthermore, the tensorial identities obtained reveal significant geometric coherence between curvature recurrence and the behavior of direction-dependent geometries. These theoretical findings are particularly relevant for geometric mechanics and theoretical physics, where Finsler-type curvature structures naturally arise in nonlinear dynamical models, anisotropic field theories, and generalized variational frameworks.

Overall, this work contributes a unified and rigorous description of curvature recurrence in Finsler spaces, laying a foundation for future research on curvature-controlled trajectories, geometric flows, and the role of Finslerian structures in physical and mechanical modeling.

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التحليل النظري لفضاءات فنسلر ثنائية التكرار من النمط العام W^h مع التركيز على الانحناء الإسقاطي لويل وأهميته في الميكانيكا الهندسية

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الملخص: يقدّم هذا العمل إطارًا نظريًا صارمًا لبنية فضاءات فنسلر ثنائية التكرار من النمط العام W^h ، مع التركيز بصورة خاصة على السلوك التحليلي لموتر الانحناء الإسقاطي لويل. ومن خلال استخدام الاشتقاق العمودي لاشتقاق كارتان واستغلال المتطابقات الحاكمة للفضاءات الفنسلرية المتصلة أفينيًا، يتم استخلاص عدة شروط مكافئة تميّز هذا النوع من الفضاءات. كما تُستنبط علاقات تبادلية جديدة للمشتقات العمودية-الاشتقاقية من الرتبة العليا لموترات الانحناء الإسقاطي، والالتواء، والانحراف. ويُوضّح التحليل كيفية تصرف هذه الموترات تحت شروط التكرار وثنائية التكرار، وكيفية انتقال هذه الخواص في السياقات المتصلة أفينيًا.

كلمات مفتاحيه: هندسة فنسلر - الانحناء الإسقاطي لويل - ثنائية التكرار W^h - الفضاءات المتصلة أفينيًا - الميكانيكا الهندسية - الفيزياء النظرية.