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ON A TWO-VARIABLE ANALOGUE HUMBERT FUNCTION

Fadhl S. N. Alsarahi
Department of Mathematics, Saber, Faculty of Science and Education, Lahej University, Yemen
Correspondence: fadhlsna@gmail.com
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Abstract: The main objective of this paper is to introduce and investigate a two-variable of the Humbert function. We derive its generating function, establish several recurrence relations, and obtain corresponding difference equations. The proposed generalization extends classical Humbert functions and provides a broader framework for analyzing special functions in both pure and applied mathematics. Potential applications in approximation theory and mathematical physics are also discussed.

Keywords: Humbert function; Two-variable and analogue; Generating function; Difference equations; Recurrence relations; Special functions.

1. Introduction: Special functions play a central role in mathematical analysis, approximation theory, and applications in physics and engineering. Among these, the Humbert function represents an important class that generalizes several classical polynomials such as Gegenbauer, Legendre, and Horadam polynomials. Although extensive research has been conducted on single-variable Humbert functions, less attention has been given to their multi-variable analogues. Recently, multi-variable extensions of Bessel and Mittag-Leffler functions have attracted significant interest [1-3]. Motivated by these developments, this paper aims to construct and analyze a two-variable analogue of the Humbert function, filling a gap in the literature and extending its theoretical framework. We present its generating function, recurrence relations, and difference equations, highlighting their potential role in future applications.

The Humbert function $J_{n,m}(x)$ is defined by means of the generating function [12]:

$$exp\left(\frac{x}{3}\left(u+t-\frac{1}{ut}\right)\right)=\sum_{n,m=-\infty}^{\infty}J_{n,m}(x)\,u^m t^n, \tag{1.1}$$

or, equivalently, by forms:

$$J_{n,m}(x)=\left(\frac{x}{3}\right)^{m+n}\frac{1}{\Gamma(m+1)\Gamma(n+1)}\ {}_0F_2\ \left(-;m+1,n+1;-\frac{x^3}{27}\right)$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(m+k+1)\Gamma(n+k+1)}\left(\frac{x}{3}\right)^{m+n+3k}. \tag{1.2}$$

Note also the immediate results

$$J_{-m,n}(x)=(-1)^m J_{m,m+n}(x)$$
$$J_{m,-n}(x)=(-1)^n J_{n+m,n}(x)\ .$$

The Bessel function of two variables is defined by the generating function [1]:

$$exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right).exp\left(\frac{yp(x)}{2}\left(w-\frac{1}{w}\right)\right)=\sum_{k,s=-\infty}^{\infty}J_{k,s}(x,y)\,t^k w^s, \tag{1.3}$$

In the next sections, we introduce a two variable of the Humbert function, some properties and study some of its recurrence relations.

2. A Two-Variable Analogue of the Humbert Function and Its Properties:

We define the Humbert function of two variables, denoted by $J_{m,n,r,s}(x,y)$, by the following

$$J_{m,n,r,s}(x,y)=J_{m,n}(x)J_{r,s}\big(yf(x)\big),$$

which can be expressed by means the following generating function:

$$exp\left(\frac{x}{3}\left(u+t-\frac{1}{ut}\right)+\frac{yf(x)}{3}\left(v+w-\frac{1}{vw}\right)\right)=exp\left(\frac{x}{3}\left(u+t-\frac{1}{ut}\right)\right).exp\left(\frac{yf(x)}{3}\left(v+w-\frac{1}{vw}\right)\right)$$
$$=\sum_{m,n,r,s=-\infty}^{\infty}J_{m,n,r,s}(x,y)\,u^m t^n v^r w^s, \tag{2.1}$$

where $x,y\in R, x,y>0, f(x)>0,\ u,t,v,w\in\mathbb{C},\ \ u,t,v,w\neq 0.$

By using well-known result, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad . \tag{2.2}$$

We obtain

$$\sum_{a,b=0}^{\infty} \sum_{c=0}^{\infty} (-1)^c \frac{1}{a!b!c!} \left(\frac{x}{3}\right)^{a+b+c} u^{a-c} t^{b-c} \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{1}{i!j!k!} \left(\frac{yf(x)}{3}\right)^{i+j+k} v^{i-k} w^{j-k}$$
$$= \sum_{m,n,r,s=-\infty}^{\infty} J_{m,n,r,s}(x,y)\, u^m t^n v^r w^s, \tag{2.3}$$

By replacing a, b, i and j by $m+c, n+c, r+k$ and $s+k$ respectively in the left-hand side of equation (2.3), we get

$$\sum_{m,n,r,s=-\infty}^{\infty} \left(\frac{x}{3}\right)^{m+n} \left(\frac{yf(x)}{3}\right)^{r+s} \sum_{c,k=0}^{\infty} (-1)^{c+k} \frac{1}{(m+c)!(n+c)!(c)!} \frac{1}{(r+k)!(s+k)!(k)!}$$
$$\times \left(\frac{x}{3}\right)^{3c} \left(\frac{yf(x)}{3}\right)^{3k} u^m t^n v^r w^s = \sum_{m,n,r,s=-\infty}^{\infty} J_{m,n,r,s}(x,y)\, u^m t^n v^r w^s, \tag{2.4}$$

by equating the coefficients of $u^m t^n v^r w^s$, we obtain the following explicit representation for a two variable of the Humbert function:

$$J_{m,n,r,s}(x,y) = \left(\frac{x}{3}\right)^{m+n} \left(\frac{yf(x)}{3}\right)^{r+s}$$
$$\times \sum_{c,k=0}^{\infty} (-1)^{c+k} \frac{1}{(m+c)!(n+c)!.c!(r+k)!(s+k)!.k!} \left(\frac{x}{3}\right)^{3c} \left(\frac{yf(x)}{3}\right)^{3k}. \tag{2.5}$$

Also, we can write

$$J_{m,n,r,s}(x,y) = \left(\frac{x}{3}\right)^{m+n} \left(\frac{yf(x)}{3}\right)^{r+s}$$
$$\times \sum_{c,k=0}^{\infty} \frac{(-1)^{c+k}}{\Gamma(m+c+1)\Gamma(n+c+1)\Gamma(c+1)\Gamma(r+k+1)\Gamma(s+k+1)\Gamma(k+1)} \left(\frac{x}{3}\right)^{3c} \left(\frac{yf(x)}{3}\right)^{3k} \quad . \tag{2.6}$$

For $y = 0$ in (2.1), we have

$$exp\left(\frac{x}{3}\left(u + t - \frac{1}{ut}\right) + \frac{0f(x)}{3}\left(v + w - \frac{1}{vw}\right)\right) = exp\left(\frac{x}{3}\left(u + t - \frac{1}{ut}\right)\right)$$
$$= \sum_{n,m=-\infty}^{\infty} J_{n,m}(x)\, u^m t^n \quad .$$

Which is the relation (1.1).

Further, it is of interest to point out that the series representation in (2.5), in particular, yields the following relationships:

$$J_{0,0,0,0}(x,y) = 1 \quad , \quad J_{m,n,0,0}(x,y) = J_{m,n}(x) \quad .$$

These corollaries demonstrate structural symmetries and parity properties of the two-variable Humbert function, which may be useful in deriving closed-form solutions for related problems.

Corollary 2.1.

If m, n, r, s be integer, then $J_{m,n,r,s}(x,y)$ satisfies the following relations:

$$J_{-m,n,r,s}(x,y) = (-1)^m J_{m,m+n,r,s}(x,y)\,, \tag{2.7}$$
$$J_{m,-n,r,s}(x,y) = (-1)^n J_{n+m,n,r,s}(x,y)\,, \tag{2.8}$$
$$J_{m,n,-r,s}(x,y) = (-1)^r J_{m,n,r,r+s}(x,y)\,, \tag{2.9}$$
$$J_{m,n,r,-s}(x,y) = (-1)^s J_{m,n,s+r,s}(x,y)\,, \tag{2.10}$$
$$J_{-m,n,-r,s}(x,y) = (-1)^{m+r} J_{m,m+n,r,r+s}(x,y)\,, \tag{2.11}$$
$$J_{m,-n,r,-s}(x,y) = (-1)^{n+s} J_{m+n,n,r+s,s}(x,y)\,, \tag{2.12}$$
$$J_{-m,-n,-r,-s}(x,y) = (-1)^{m+r} J_{m,m-n,r,r-s}(x,y)\,. \tag{2.13}$$

Proof. Using (2.5), we get

$$J_{-m,n,r,s}(x,y) = \sum_{c=m}^{\infty} \sum_{k=0}^{\infty} (-1)^{c+k} \frac{1}{(-m+c)!(n+c)!c!(r+k)!(s+k)!k!}$$
$$\times \left(\frac{x}{3}\right)^{-m+n+3c} \left(\frac{yf(x)}{3}\right)^{r+s+3k} , \tag{2.14}$$

by replacing c with $c+m$ in the r. h. s. of (2.14), we obtain

$$J_{-m,n,r,s}(x,y;p,q)$$
$$= \sum_{c,k=0}^{\infty} (-1)^{c+m+k} \frac{1}{c!(m+n+c)!(m+c)!(r+k)!(s+k)!k!}$$

$$\times\left(\frac{x}{3}\right)^{2m+n+3c}\left(\frac{yf(x)}{3}\right)^{r+s+3k}, \tag{2.15}$$

which on using definition (2.5) gives yields the required relation (2.7).
Similarly, we can prove the relations (2.8), (2.9) and (2.10).
Also,

$$J_{-m,n,-r,s}(x,y)=\sum_{c=m}^{\infty}\sum_{k=r}^{\infty}(-1)^{c+k}\frac{1}{(-m+c)!(n+c)!.c!(-r+k)!(s+k)!k!}$$
$$\times\left(\frac{x}{3}\right)^{-m+n+3c}\left(\frac{yf(x)}{3}\right)^{-r+s+3k}, \tag{2.16}$$

replacing c by $c+m$ and k by $k+r$ in the r. h. s. of (2.16), we find

$$J_{-m,n,-r,s}(x,y)=\sum_{c,k=0}^{\infty}(-1)^{c+m+k+r}\frac{1}{c!(m+n+c)!(m+c)!(k)!(r+s+k)!(r+k)!}$$
$$\times\left(\frac{x}{3}\right)^{2m+n+3c}\left(\frac{yf(x)}{3}\right)^{2r+s+3k}=(-1)^{m+r}J_{m,m+n,r,r+s}(x,y;p,q)$$

which on using definition (2.5) gives yields the required relation (2.11).
The equation (2.12) can be proved in a like manner.
By a similar analysis, it can be shown that

$$J_{-m,-n,-r,-s}(x,y)=\sum_{c=\max(m,n)}^{\infty}\sum_{k=\max(r,s)}^{\infty}(-1)^{c+k}\frac{1}{(-m+c)!(-n+c)!(c)!}$$
$$\times\frac{1}{(-r+k)!(-s+k)!(k)!}\left(\frac{x}{3}\right)^{-m-n+3c}\left(\frac{yf(x)}{3}\right)^{-r-s+3k}, \tag{2.17}$$

Upon setting c by $c+m$ and k by $k+r$ in the r. h. s. of (2.17), we deduce

$$J_{-m,-n,-r,-s}(x,y)$$
$$=\sum_{c,k=0}^{\infty}(-1)^{c+m+k+r}\frac{1}{c!(m-n+c)!(m+c)!(k)!(r-s+k)!(r+k)!}$$
$$\times\left(\frac{x}{3}\right)^{2m-n+3c}\left(\frac{yf(x)}{3}\right)^{2r-s+3k}=(-1)^{m+r}J_{m,m-n,r,r-s}(x,y;p,q),$$

from using definition (2.5) gives yields the required relation (2.13).
Also, upon setting c by $c+n$ and k by $k+s$ in the r. h. s. of (2.17), we get

$$J_{-m,-n,-r,-s}(x,y)=(-1)^{n+s}J_{n-m,n,s-r,s}(x,y). \tag{2.18}$$

Corollary 2.2. If c,k are even numbers and $f(x)$ is an even function, then the function $J_{m,n}(x,y)$ satisfies the relations:

$$J_{m,n,r,s}(-x,y)=(-1)^{m+n}J_{m,n,r,s}(x,y), \tag{2.19}$$
$$J_{m,n,r,s}(x,-y)=(-1)^{r+s}J_{m,n,r,s}(x,y), \tag{2.20}$$
$$J_{m,n,r,s}(-x,-y)=(-1)^{m+n+r+s}J_{m,n,r,s}(x,y). \tag{2.21}$$

Proof. Since,

$$J_{m,n,r,s}(-x,y)=\left(-\frac{x}{3}\right)^{m+n}\left(\frac{yf(-x)}{3}\right)^{r+s}$$
$$\times\sum_{c,k=0}^{\infty}(-1)^{c+k}\frac{1}{(m+c)!(n+c)!(c)!(r+k)!(s+k)!(k)!}\left(-\frac{x}{3}\right)^{3c}\left(\frac{yf(-x)}{3}\right)^{3k}$$
$$=(-1)^{m+n}\left(\frac{x}{3}\right)^{m+n}\left(\frac{yf(x)}{3}\right)^{r+s}$$
$$\times\sum_{c,k=0}^{\infty}(-1)^{c+k}\frac{1}{(m+c)!(n+c)!(c)!(r+k)!(s+k)!(k)!}\left(\frac{x}{3}\right)^{3c}\left(\frac{yf(x)}{3}\right)^{3k},$$

which in view of (2.5), becomes relation (2.19).
Similarly, the relations (2.20) and (2.21), can be proved.

3. Difference Equatios

In this section, we establish difference equations satisfied by the two-variable Humbert function. These relations provide an effective tool for recursive computation and can be compared with analogous relations for Bessel and Legendre functions.

Theorem (3.1). The Humbert functions $J_{m,n,r,s}(x,y)$ satisfy the following relation:

$$\frac{\partial}{\partial x}J_{m,n,r,s}(x,y)=\frac{1}{3}\left(J_{m-1,n,r,s}(x,y)+J_{m,n-1,r,s}(x,y)-J_{m+1,n+1,r,s}(x,y)\right)$$
$$+\frac{yf^{/}(x)}{3}\left(J_{m,n,r-1,s}(x,y)+J_{m,n,r,s-1}(x,y)-J_{m,n,r+1,s+1}(x,y)\right), \tag{3.1}$$

and

$$\frac{\partial}{\partial y}J_{m,n,r,s}(x,y)=\frac{f(x)}{3}\Big(J_{m,n,r-1,s}(x,y)+J_{m,n,r,s-1}(x,y)-J_{m,n,r+1,s+1}(x,y)\Big). \tag{3.2}$$

Proof. Differentiating (2.1) with respect to x, we obtain

$$\sum_{m,n,r,s=-\infty}^{\infty}\frac{\partial}{\partial x}J_{m,n,r,s}(x,y)\,u^m t^n v^r w^s$$
$$=\frac{1}{3}\Big(u+t-\frac{1}{ut}\Big)\,exp\left(\frac{x}{3}\Big(u+t-\frac{1}{ut}\Big)\right).exp\left(\frac{yf(x)}{3}\Big(v+w-\frac{1}{vw}\Big)\right)$$
$$+\frac{yf^{/}(x)}{3}\Big(v+w-\frac{1}{vw}\Big)\,exp\left(\frac{x}{3}\Big(u+t-\frac{1}{ut}\Big)\right).exp\left(\frac{yf(x)}{3}\Big(v+w-\frac{1}{vw}\Big)\right), \tag{3.2}$$

and using (2.2), yields

$$\sum_{m,n,r,s=-\infty}^{\infty}\frac{\partial}{\partial x}J_{m,n,r,s}(x,y)\,u^m t^n v^r w^s=\frac{1}{3}\Big(u+t-\frac{1}{ut}\Big)\sum_{a,b=0}^{\infty}\sum_{c=0}^{\infty}(-1)^c\frac{1}{a!b!c!}$$
$$\times\Big(\frac{x}{3}\Big)^{a+b+c}u^{a-c}t^{b-c}\sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{1}{i!j!k!}\Big(\frac{yf(x)}{3}\Big)^{i+j+k}v^{i-k}w^{j-k}$$
$$+\frac{yf^{/}(x)}{3}\Big(v+w-\frac{1}{vw}\Big)\sum_{a,b=0}^{\infty}\sum_{c=0}^{\infty}(-1)^c\frac{1}{a!b!c!}\Big(\frac{x}{3}\Big)^{a+b+c}u^{a-c}t^{b-c}$$
$$\times\sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{1}{i!j!k!}\Big(\frac{yf(x)}{3}\Big)^{i+j+k}v^{i-k}w^{j-k}. \tag{3.3}$$

By substituting a,b,i and j by $m+c,n+c,r+k$ and $s+k$ respectively in the right-hand side of equation (3.3), we see that

$$\sum_{m,n,r,s=-\infty}^{\infty}\frac{\partial}{\partial x}J_{m,n,r,s}(x,y)\,u^m t^n v^r w^s=\frac{1}{3}\Big(u+t-\frac{1}{ut}\Big)\sum_{m,n,r,s=-\infty}^{\infty}\Big(\frac{x}{3}\Big)^{m+n}\Big(\frac{yf(x)}{3}\Big)^{r+s}$$
$$\times\sum_{c,k=0}^{\infty}\frac{(-1)^{c+k}}{(m+c)!(n+c)!(c)!(r+k)!(s+k)!(k)!}\Big(\frac{x}{3}\Big)^{3c}\Big(\frac{yf(x)}{3}\Big)^{3k}u^m t^n v^r w^s$$
$$+\frac{yf^{/}(x)}{3}\Big(v+w-\frac{1}{vw}\Big)\sum_{m,n,r,s=-\infty}^{\infty}\Big(\frac{x}{3}\Big)^{m+n}\Big(\frac{yf(x)}{3}\Big)^{r+s}$$
$$\times\sum_{c,k=0}^{\infty}\frac{(-1)^{c+k}}{(m+c)!(n+c)!(c)!(r+k)!(s+k)!(k)!}\Big(\frac{x}{3}\Big)^{3c}\Big(\frac{yf(x)}{3}\Big)^{3k}u^m t^n v^r w^s. \tag{3.4}$$

By equating the coefficients of $t^m w^n v^r w^s$ in (3.4), we get relation (3.1).
In a similar manner, we can prove (3.2).

4. Recurrence Relations: Recurrence relations are central to both theoretical analysis and numerical computation of special functions. For the two-variable Humbert function, the following recurrence relations hold.

Theorem (4.1). The polynomials sequence $J_{m,n,r,s}(x,y)$ satisfies the next recurrence relations

$$(m+1)\,J_{m+1,n,r,s}(x,y)=\frac{x}{3}\Big(J_{m,n,r,s}(x,y)+J_{m+2,n+1,r,s}(x,y)\Big), \tag{4.1}$$

$$(n+1)J_{m,n+1,r,s}(x,y)=\frac{x}{3}\Big(J_{m,n,r,s}(x,y)+J_{m+1,n+2,r,s}(x,y)\Big), \tag{4.2}$$

$$(r+1)\,J_{m,n,r+1,s}(x,y)=\frac{yf(x)}{3}\Big(J_{m,n,r,s}(x,y)+J_{m,n,r+2,s+1}(x,y)\Big), \tag{4.3}$$

and

$$(s+1)\,J_{m,n,r,s+1}(x,y)=\frac{yf(x)}{3}\Big(J_{m,n,r,s}(x,y)+J_{m,n,r+1,s+2}(x,y)\Big). \tag{4.4}$$

Proof. Differentiating (2.5) with respect to u, we find

$$\sum_{m,n,r,s=-\infty}^{\infty}\frac{\partial}{\partial u}J_{m,n,r,s}(x,y;p,q)\,u^m t^n v^r w^s$$
$$=\frac{x}{3}exp\Big[\frac{xu}{3}\Big].exp\Big[\frac{xt}{3}\Big].exp\Big[-\frac{x}{3ut}\Big].exp\Big[\frac{yf(x)v}{3}\Big].exp\Big[\frac{yf(x)w}{3}\Big].exp\Big[-\frac{yf(x)}{3vw}\Big]$$
$$+\frac{x}{3u^2t}exp\Big[\frac{xu}{3}\Big].exp\Big[\frac{xt}{3}\Big].exp\Big[-\frac{x}{3ut}\Big].exp\Big[\frac{yf(x)v}{3}\Big].exp\Big[\frac{yf(x)w}{3}\Big].exp\Big[-\frac{yf(x)}{3vw}\Big]$$

By using relation (2.2), we obtain

$$\sum_{m,n,s,r=-\infty}^{\infty}m\,J_{m,n,r,s}(x,y)u^{m-1}t^n v^r w^s=\frac{x}{3}\sum_{a,b=0}^{\infty}\sum_{c=0}^{\infty}(-1)^c\frac{1}{a!b!c!}\Big(\frac{x}{3}\Big)^{a+b+c}u^{a-c}t^{b-c}$$
$$\times\sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{1}{i!j!k!}\Big(\frac{yf(x)}{3}\Big)^{i+j+k}v^{i-k}w^{j-k}$$
$$+\frac{x}{3u^2t}\sum_{a,b=0}^{\infty}\sum_{c=0}^{\infty}(-1)^c\frac{1}{a!b!c!}\Big(\frac{x}{3}\Big)^{a+b+c}u^{a-c}t^{b-c}$$

$$\times \sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{1}{i!j!k!}\left(\frac{yf(x)}{3}\right)^{i+j+k} v^{i-k}w^{j-k} . \tag{4.5}$$

By replacing a, b, i and j by $m+c, n+c, r+k$ and $s+k$ respectively in the right-hand side of equation (4.5), we get

$$\sum_{m,n,s,r=-\infty}^{\infty}(m+1)\, J_{m+1,n,r,s}(x,y;p,q)u^m t^n v^r w^s$$
$$= \frac{x}{3}\sum_{m,n=-\infty}^{\infty}\sum_{c=0}^{\infty}(-1)^c \frac{1}{(m+c)!(n+c)!c!}\left(\frac{x}{3}\right)^{m+n+3c} u^m t^n$$
$$\times \sum_{r,s=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{1}{(r+k)!(s+k)!k!}\left(\frac{yf(x)}{3}\right)^{r+s+3k} v^r w^s$$
$$+ \frac{x}{3u^2t}\sum_{m,n=-\infty}^{\infty}\sum_{c=0}^{\infty}(-1)^c \frac{1}{[m+c]!(n+c)!c!}\left(\frac{x}{3}\right)^{m+n+3c} u^m t^n$$
$$\times \sum_{r,s=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{1}{(r+k)!(r+k)!k!}\left(\frac{yf(x)}{3}\right)^{r+s+3k} v^r w^s .$$

Comparing of both sides, we get the relation (4.1).

Similarly, way differentiating (2.5) with respect to t, v and w, we find relations (4.2), (4.3) and (4.4) respectively.

Theorem (4.2). A two variable of the Humbert function $J_{m,n,r,s}(x,y)$ satisfies the following relation:

$$\frac{1}{m!n!r!s!}\left(\frac{x}{3}\right)^{m+n}\left(\frac{yf(x)}{3}\right)^{r+s} = \sum_{c,k=0}^{\infty}\frac{1}{c!k!}\left(\frac{x}{3}\right)^{c}\left(\frac{yf(x)}{3}\right)^{k} J_{m+c,n+c,r+k,s+k}(x,y) . \tag{4.6}$$

Proof. Using generating function of function $J_{m,n,r,s}(x,y)$ and definition expression for functions

$exp\left[\frac{xu}{3}\right], exp\left[\frac{xt}{3}\right], exp\left[\frac{yf(x)v}{3}\right], exp\left[\frac{yf(x)w}{3}\right], exp\left[\frac{x}{3ut}\right]$ and $exp\left[\frac{yf(x)}{3vw}\right]$ we have

$$exp\left[\frac{xu}{3}\right].exp\left[\frac{xt}{3}\right].exp\left[\frac{yf(x)v}{3}\right].exp\left[\frac{yf(x)w}{3}\right]$$
$$= exp\left[\frac{x}{3ut}\right].exp\left[\frac{yf(x)}{3vw}\right]\sum_{m,n,s,r=-\infty}^{\infty} J_{m,n,r,s}(x,y)\, u^m t^n v^r w^s .$$

Using the relation (2.2)

$$\sum_{m,n,r,s=0}^{\infty}\frac{1}{m!n!r!s!}\left(\frac{x}{3}\right)^{m+n}\left(\frac{yf(x)}{3}\right)^{r+s} u^m t^n v^r w^s$$
$$= \sum_{c,k=0}^{\infty}\frac{1}{c!k!}\left(\frac{x}{3}\right)^{c}\left(\frac{yf(x)}{3}\right)^{k}\sum_{m,n,s,r=-\infty}^{\infty} J_{m,n,r,s}(x,y)\, u^{m-c}t^{n-c}v^{r-k}w^{s-k}$$
$$= \sum_{c,k=0}^{\infty}\frac{1}{c!k!}\left(\frac{x}{3}\right)^{c}\left(\frac{yf(x)}{3}\right)^{k}\sum_{m,n,s,r=0}^{\infty} J_{m+c,n+c,r+k,s+k}(x,y)\, u^m t^n v^r w^s .$$

Comparing of the coefficients of $u^m t^n v^r w^s$ of the above equation, we obtain the required relation (4.6).

Conclusion: In this work, we have introduced a two-variable analogue of the Humbert function and derived several of its fundamental properties including generating functions, recurrence relations, and difference equations. This generalization extends the classical Humbert function and enriches the theory of special functions. Future work may focus on exploring applications in mathematical physics, approximation theory, and numerical analysis, as well as extending the framework to fractional and q-analogues.

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حول دالة همبرت المتناظرة لمتغيرين

فضل صالح ناصر السرحي
قسم الرياضيات – كلية صبر للعلوم والتربية – جامعة لحج – اليمن

الملخص: الهدف الرئيسي من هذه الورقة هو تقديم والتحقق من دالة همبرت لمتغيرين ونشتق دالة التوليد الخاصة به. ونحصل على معادلات الفرق المقابلة وننشئ العديد من العلاقات التكرارية. يوسع التعميم المقترح دولة همبرت الكلاسيكية ويوفر إطاراً أوسع لتحليل الدوال الخاصة في كل من الرياضيات البحتة والتطبيقية. كما تمت مناقشة التطبيقات المحتملة في نظرية التقريب والفيزياء الرياضية.

الكلمات المفتاحية: دالة همبرت المتناظرة لمتغيرين، الدالة المولدة، معادلات الفرق، العلاقات التكرارية، الدوال الخاصة.