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Expansion of Curvature Tensors under the Berwald Covariant Derivative in Finsler Spaces

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Abstract: Curvature tensors are fundamental tools in differential geometry for describing the geometric structure of manifolds. In this paper, we study the expansion of several curvature tensors in Finsler spaces under the Berwald covariant derivative. Various curvature tensors, including the Riemannian, projective, conformal, conharmonic, concircular, and P_1-curvature tensors, are expressed in terms of the Weyl projective curvature tensor. A generalized expansion formula is derived, and a number of identities and theorems concerning the Berwald covariant derivatives of these curvature tensors are established. The obtained results extend classical relations from Riemannian geometry to the Finslerian setting and contribute to the study of curvature structures in Finsler manifolds.

Keywords: Riemannian curvature tensor; Berwald covariant derivative; Weyl projective curvature tensor; Conformal curvature tensor; Conharmonic curvature tensor; Concircular curvature tensor.

1. Introduction

Curvature is one of the most fundamental concepts in differential geometry, providing a precise mathematical description of the geometric and topological properties of manifolds. The Riemannian curvature tensor serves as the principal tool for measuring curvature in Riemannian manifolds and plays a crucial role in several branches of mathematics and physics, particularly in general relativity and gravitational theory. Over time, various curvature tensors have been introduced to capture different geometric features, including the projective, conformal, conharmonic, concircular, and P_1-curvature tensors. These tensors are closely related to the Riemannian curvature tensor and often arise from specific geometric transformations or invariance properties.

Finsler geometry, as a natural generalization of Riemannian geometry, allows the metric to depend not only on position but also on direction. This additional flexibility leads to richer geometric structures and requires the development of new tools for studying curvature. Among these tools, the Berwald covariant derivative plays a significant role in analyzing the behavior of tensor fields on Finsler manifolds.The present work is devoted to the study of the expansion of curvature tensors in Finsler spaces using the Berwald covariant derivative. We begin by reviewing the definitions and properties of several important curvature tensors and expressing them in terms of the Weyl projective curvature tensor. A generalized expansion curvature tensor is then introduced, and its expansion formula is derived. Subsequently, we investigate the expansion of various curvature tensors by establishing identities and proving a sequence of theorems that describe their behavior under the Berwald covariant derivative. These results generalize several known formulas from Riemannian geometry and provide new insights into the structure of curvature tensors in Finsler geometry. Curvature tensors constitute a fundamental aspect of differential geometry and play an essential role in both Riemannian and Finsler geometries. In the Riemannian setting, curvature tensors have been widely applied to the geometric formulation of spacetime and general relativity, as discussed by Ahsan and Ali (2016).

Finsler geometry extends Riemannian geometry by allowing direction-dependent metrics, leading to richer curvature structures. Classical contributions by Matsumoto (1971) and Rund (1981) established the foundational theory of curvature and recurrence in Finsler spaces. The notion of recurrent manifolds, originally introduced by Chaki (1970), was later generalized to concircular and higher-order recurrent structures by Maralebhavi and Rathnamma (1999).

Subsequent studies focused on generalized and higher-order recurrent Finsler spaces, employing various curvature tensors and geometric connections. Notable contributions include works by Mishra and Lodhi (2008), Misra et al. (2014), and Pandey et al. (2011), who investigated recurrence conditions using Berwald curvature and higher-order derivatives.

More recent research by Al-Qashbari and collaborators has emphasized generalized curvature tensors, Weyl-type tensors, and their behavior under Berwald's and Cartan's covariant derivatives. These studies addressed recurrence, decomposition, and higher-order generalizations of curvature tensors in Finsler spaces. However, a systematic treatment of expansion properties of curvature tensors under the Berwald covariant derivative remains limited. The present work aims to address this gap by establishing expansion formulas and related identities for several important curvature tensors in Finsler geometry.

The derivative $\mathcal{B}_m \mathrm{T}_\mathrm{j}^\mathrm{i}$ for Berwald's $(\mathcal{B}_m)$ of any tensor $\mathrm{T}_\mathrm{j}^\mathrm{i}$, w. r. t. x^m is defined as

$$\mathcal{B}_m T_j^i = \partial_m T_j^i - \left(\dot{\partial}_r T_j^i\right) G_m^r + T_j^r G_{rm}^i - T_r^i \, G_{jm}^r \; . \qquad (1.1)$$

[1]

The vector y^i and metric function F are vanished identically for Berwald's covariant derivative.

$$\text{(a)} \quad \mathcal{B}_m \mathrm{F} = 0 \quad \text{and} \quad \text{(b)} \quad \mathcal{B}_m \, y^i = 0 \, . \qquad (1.2)$$

The metric tensor g_{ij} is not equal to zero (i.e. not vanish) for Berwald's covariant derivative

$$\mathcal{B}_k g_{ij} = -2 \, C_{ijk|h} \, y^h = -2 \, y^h \mathcal{B}_h C_{ijk} \; . \qquad (1.3)$$

The quantities g_{ij} and g^{ij} are related by

$$\text{(a)} \quad g_{ij} \, g^{jk} = \delta_i^k = \begin{cases} 1 \;, & if \quad i = k \;, \\ 0 \;, & if \quad i \neq k \;. \end{cases} \quad \text{and} \quad \text{(b)} \quad g_{ij} \, y^i = y_j \; . \qquad (1.4)$$

The tensors R_{jkh}^i and H_{jk}^i give the following identities

$$\text{a)} \;\; R_{jkh}^i \, y^j = H_{kh}^i \;, \;\; \text{b)} \;\; H_{kh}^i \, y^k = H_h^i \;\; \text{and c)} \;\; H_{ki}^i = H_k \;\; \text{and d)} \;\; H_i^i = (n-1)H \; . \qquad (1.5)$$

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[1] $\partial_i = \frac{\partial}{\partial x^i}$; $\dot{\partial}_i = \frac{\partial}{\partial y^i}$

Covarian derivative of $\mathcal{B}_m R_{ij}$, of Ricci tensor R_{ij} and $\mathcal{B}_m\, \delta_i^k$ given by

$$a)\ \mathcal{B}_m R_{ij} = \lambda_m R_{ij} \quad \text{and} \quad b)\ \mathcal{B}_m\, \delta_i^k = 0\,. \tag{1.6}$$

Also, Covarian derivative of $\mathcal{B}_m$, we have

$$a)\ \mathcal{B}_m \delta_h^k\, R_{ij} = \lambda_m \delta_h^k\, R_{ij} \quad , \quad b)\ \mathcal{B}_m\, g_{ij}\, R_h^k = \lambda_m\, g_{ij}\, R_h^k \ ,$$

$$c)\ \mathcal{B}_m R \delta_k^h g_{ij} = \lambda_m R \delta_k^h g_{ij} \ \text{and} \quad d)\ \mathcal{B}_m R R_{ij} = \lambda_m R R_{ij} \ . \tag{1.7}$$

A large number of researchers have presented the following identities in their works

$$a)\ \ C_{ijk}\, y^i = 0 \ \ , \ b)\ \ C_{ijk} = \tfrac{1}{4}\left(\dot\partial_k \dot\partial_i \dot\partial_j F^2\right) \ \ , \ c)\ \ \dot\partial_j\, y^j = 1 \ \ , \ d)\ \ y_j\, y^j = F^2 \ ,$$

$$e)\ \ \delta_j^k\, y^j = y^k \ \ , \ f)\ \ y_j\, y^j = F^2 \quad \text{and} \quad h)\ \ \dot\partial_k\, y_j = g_{jk} \ . \tag{1.8}$$

Derivative for Berwald's $(\mathcal{B}_m)$ of the tensors T_{ijk}^h , T_{ij}^h and T_i^h, w. r. t. x^m are defined as

$$a)\ \ \mathcal{B}_m T_{ijk}^h = \lambda_m T_{ijk}^h \ \ , \ b)\ \mathcal{B}_m T_{ij}^h = \lambda_m T_{ij}^h \quad \text{and} \quad c)\ \ \mathcal{B}_m T_i^h = \lambda_m T_i^h \ . \tag{1.9}$$

The plan of the present paper is as follows: After the section 1 introduction and preliminaries, we study expansion for any curvature tensor with respect to Berwald covariant derivative. Section 2 gives the relationships between Weyl projective curvature tensor and some others curvature tensors. Section 3 study an expansion of Berwald covariant derivative for any curvature tensor. In last section we investigation the identities that were given in section 2 by using the expansion.

2. Preliminaries

There is a relationship between any two curvature tensors in Finsler geometry, this relationship is showed by a mathematical identity, here we will discuss the relationship between Weyl projective curvature tensor and the following curvature tensors:

2.1. The Riemanniian Curvature Tensor R_{jkh}^i

In the mathematical field of differential geometry, the Riemann curvature tensor is the most common way used to express the curvature of Riemannian manifold. It assigns a tensor to each point of a Riemannian manifold (i.e., it is a tensor field). It is a local invariant of Riemannian metrics which measures the failure of the second covariant derivatives to commute. A Riemannian manifold has zero curvature if and only if it is flat, i.e. locally isometric to the Euclidean space. The curvature tensor can also be defined for any pseudo-Riemannian manifold, or indeed any manifold equipped with an affine connection. The Riemann curvature tensor is a tool used to describe the curvature of n-dimensional spaces such as Riemannian manifold in the field of differential geometry.

The Riemann curvature tensor plays an important role in the theories of general relativity and gravity as well as the curvature of a spacetime. It is closely related to the Weyl projective curvature tensor.

Definition 2.1. Weyl projective curvature tensor in terms of Riemannian curvature tensor R_{jkh}^i is defined as [11] and [18].

$$W_{jkh}^i = R_{jkh}^i + \frac{1}{(n-1)}\left(\delta_k^i R_{jh} - R_h^i g_{jk}\right) \ . \tag{2.1}$$

In (V_4, F), we have

$$R^i_{jkh} = W^i_{jkh} - \frac{1}{3}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) . \tag{2.2}$$

The tensors W^i_{jkh} and W^i_{jk} give the following identities

$$\text{a) } W^i_{jkh}\, y^j = W^i_{kh} \ , \ \text{b) } W^i_{jk}\, y^j = W^i_k \ , \ \text{c) } W^i_{jk}\, y^j = 0 \ \text{ and } \ \text{d) } W^i_i = 0 . \tag{2.3}$$

2.2. Projective Curvature Tensor $\bar{W}^i_{jkh}$

The $\bar{W}$-projective curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conharmonic curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology.

The properties of an M-projective curvature tensor were proposed by Pokhariyal and Mishra in 1970. This tensor is described as follows

$$\bar{W}(X,Y,Z,T) = \bar{R}(X,Y,Z,T) - \frac{1}{2(n-1)}[S(Y,Z)g(X,T) - S(X,Z)g(Y,T)$$
$$+ g(Y,Z)S(X,T) - g(X,T)S(Y,Z)] . \tag{2.4}$$

Where: $\bar{W}(X,Y,Z,T) = g(W(X,Y)Z,T)$ and $\bar{R}(X,Y,Z,T) = g(R(X,Y)Z,T)$.

R is the Riemann curvature tensor, S is the Ricci tensor, g is the metric tensor, n is the dimension of the manifold. The $\bar{W}$-projective curvature tensor has a number of interesting properties. For example, it is invariant under conformal transformations. This means that it is the same for two metrics that are conformally equivalent. The $\bar{W}$-projective curvature tensor also vanishes if and only if the manifold is Ricci-flat. The $\bar{W}$-projective curvature tensor has been used to study a variety of geometric problems. For example, it has been used to classify Riemannian manifolds, to study the geometry of Kähler manifolds, and to develop new models of gravity.

The local coordinates expression of equation (2.4) as follows

$$\bar{W}_{ljkh} = R_{ljkh} - \frac{1}{2(n-1)}\left(R_{jk}g_{lh} - R_{lk}g_{jh} + g_{jk}R_{lh} - g_{lk}R_{jh}\right) . \tag{2.5}$$

Assuming $n = 4$ and using (2.2) in equation (2.5) and contracting with g^{li}, the M-projective curvature tensor is given by

$$\bar{W}^i_{jkh} = W^i_{jkh} - \frac{1}{6}\left(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk}R^i_h - g_{jh}R^i_k\right). \tag{2.6}$$

2.3. Conformal Curvature Tensor C^i_{jkh}

The conformal curvature tensor, also known as the Weyl conformal curvature tensor, is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force.

Definition 2.2. The Conformal curvature tensor C^h_{ijk} expressed as follows

$$C^i_{jkh} = R^i_{jkh} - \tfrac{1}{2}\left(\delta^i_k R_{jh} - \delta^i_h R_{jk} + R^i_k g_{jh} - R^i_h g_{jk}\right) - \tfrac{1}{6} R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right). \tag{2.7a}$$

Using (2.2) in equation (2.7a), we get

$$C^i_{jkh} = W^i_{jkh} - \tfrac{5}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \tfrac{1}{6} R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right). \tag{2.7b}$$

2.4. Conharmonic Curvature Tensor L^i_{jkh}

The conharmonic curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conformal curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology.

Definition 2.3. For V_4 the Conharmonic curvature tensor L^i_{jkh} defined as

$$L^i_{jkh} = R^i_{jkh} - \tfrac{1}{2}\left(g_{jk} R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh} R^i_k\right). \tag{2.8a}$$

Using (2.2) in equation (2.8a), we get

$$L^i_{jkh} = W^i_{jkh} + \tfrac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \tfrac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right). \tag{2.8b}$$

2.5. Concircular Curvature Tensor M^i_{jkh}

The concircular curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the conformal curvature tensor (also known as the Weyl curvature tensor) and the projective curvature tensor. The concircular curvature tensor vanishes if and only if the manifold is concircularly flat.

Definition 2.4. The Concircular curvature tensor M_{hijk}, for V_4 is defined as

$$M^i_{jkh} = R^i_{jkh} - \tfrac{1}{12} R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right). \tag{2.9}$$

Using (2.2) in equation (2.9), we get

$$M^i_{jkh} = W^i_{jkh} - \tfrac{1}{12} R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right) - \tfrac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right). \tag{2.10}$$

2.6. P_1- Curvature Tensor

The P_1-curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the Ricci curvature tensor and the scalar curvature. The P_1-curvature tensor vanishes if and only if the manifold is Ricci-flat and has constant scalar curvature. The tensor $P_1(X,Y,Z,T)$ has been defined (Pokhariyal 1973), as

$$P_1(X,Y,Z,T) = R(X,Y,Z,T) + \tfrac{1}{2(n-1)}[\, g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z)$$
$$-g(X,Z)Ric(Y,T) + g(X,T)Ric(Y,Z)]. \tag{2.11}$$

We consider the P_1-curvature tensor in the index notation as

$$P_{1_{ljkh}} = R_{ljkh} + \tfrac{1}{2(n-1)}\left(g_{jk}R_{lh} - g_{jh}R_{lk} - g_{lk}R_{jh} + g_{lh}R_{jk}\right). \tag{2.12}$$

This can be written as

$$P^i_{1_{jkh}} = R^i_{jkh} + \tfrac{1}{2(n-1)}\left(g_{jk}R^i_h - g_{jh}R^i_k - \delta^i_k R_{jh} + \delta^i_h R_{jk}\right). \tag{2.13}$$

In (V_4, F), and using (2.2) in equation (2.13), we get

$$P^i_{1jkh} = W^i_{jkh} + \frac{1}{6}\left(\delta^i_h R_{jk} - g_{jh} R^i_k\right) - \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) . \quad (2.14)$$

3. Expansion Curvatures Tensors in Finsler Space

The expansion curvature tensor T is a geometric object introduced in Finsler geometry. It is a measure of the curvature of a Finsler manifold, which is a generalization of a Riemannian manifold. The expansion curvature tensor is closely related to the Riemann curvature tensor and the Berwald curvature tensor. It vanishes if and only if the Finsler manifold is flat. we introduced the generalized by Berwald covariant derivative β_m for any tensor T^h_{ijk} was given by

$$\beta_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) . \quad (3.1)$$

From (1.8), we can write (3.1) by the follows form

$$\beta_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \left[W^i_h C_{ijk} y^i - W^i_k C_{ijh} y^i\right] . \quad (3.2)$$

Using (1.9) in (3.2), we get

$$\beta_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left(W^i_h \dot{\partial}_k \dot{\partial}_i \dot{\partial}_j F^2 y^i - W^i_k \dot{\partial}_h \dot{\partial}_j \dot{\partial}_i F^2 y^i\right) . \quad (3.3)$$

From (1.11), applying (1.10) on (3.3), we get

$$\beta_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left(W^i_h \dot{\partial}_k \dot{\partial}_j y^j y_j - W^i_k \dot{\partial}_h \dot{\partial}_j y^j y_j\right) . \quad (3.4)$$

Applying (1.10) again on (3.4), we get

$$\beta_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left(W^i_h \dot{\partial}_k y_j - W^i_k \dot{\partial}_h y_j\right) .$$

From (1.12), we have

$$\beta_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) . \quad (3.5)$$

From the previous steps, we can conclude the following theorem

Theorem3.1. The expansion of (1.13) is given by (3.5).

The dimensionality of many curvatures tensors operators will be extended in accordance with theorem 3.1.

4. Investigate the Expansion by Identities

Mathematical identities are equations that are always true, regardless of the values of the variables involved. They can be used to simplify expressions, solve equations, and prove theorems. we investigated the expansion of Berwald covariant derivative for any curvature tensor that was given in (3.5), i.e.

$$\beta_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) . \quad (4.1)$$

We suppose that (4.1) holds to investigate the following identities

4-1. By tack away Berwald covariant derivative for (2.2), we have

$$\beta_m R^i_{jkh} = \beta_m W^i_{jkh} - \frac{1}{3}\beta_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) . \quad (4.2)$$

From (1.7a), (4.1) and (4.2), we get

$$\beta_m R^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) - \frac{1}{3}\lambda_m(\delta^i_k R_{jh} - R^i_h g_{jk}) .$$

This give

$$\beta_m R^i_{jkh} = \lambda_m \left(W^i_{jkh} - \frac{1}{3}(\delta^i_k R_{jh} - R^i_h g_{jk})\right) + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}). \quad (4.3)$$

By using (2.2) in (4.3), we have

$$\beta_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) . \quad (4.4)$$

From the previous steps, we can conclude the following theorem

Theorem 4.1: The expansion derivative for Berwald of Riemannian curvature tensor R^i_{jkh} (2.2) is satisfies the equation (4.4).

Transvecting the condition to a higher dimensional space (4.4) by y^j, using (1.2b), (1.5a) and (1.4b), we get

$$\beta_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \frac{1}{4}(W^i_h y_k - W^i_k y_h) . \quad (4.5)$$

Again, transvecting condition to a higher dimensional space (4.5) by y^k, using (1.2b), (1.5b), (2.3b), (1,8d) and (1.8e), we get

$$\beta_m H^i_h = \lambda_m H^i_h + \mu_m(\delta^i_h F^2 - y^i y_h) + \frac{1}{4} W^i_h F^2 . \quad (4.6)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.2. In covariant derivative for Berwald of first order for torsion tensor H^i_{kh} and deviation tensor H^i_h are given by (4.5) and (4.6).

4-2. Tack away Berwald covariant derivative for (2.6), we have

$$\beta_m \bar{W}^i_{jkh} = \beta_m W^i_{jkh} - \frac{1}{6}\beta_m(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk} R^i_h - g_{jh} R^i_k) . \quad (4.7)$$

From (1.7a), (1.7b), (4.1) and (4.7), we get

$$\beta_m \bar{W}^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh})$$
$$- \frac{1}{6}\lambda_m(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk} R^i_h - g_{jh} R^i_k) .$$

This can be written as

$$\beta_m \bar{W}^i_{jkh} = \lambda_m \left(W^i_{jkh} - \frac{1}{6}(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk} R^i_h - g_{jh} R^i_k)\right)$$
$$+ \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) . \quad (4.8)$$

From (2.6) and (4.8), we have

$$\beta_m \bar{W}^i_{jkh} = \lambda_m \bar{W}^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) . \quad (4.9)$$

So, the proof of theorem is completed, we can say

Theorem 4.3. The expansion derivative for Berwald of projective curvature tensor $\bar{W}^i_{jkh}$ (2.6) is satisfies the equation (4.9).

4-3. Tack away Berwald covariant derivative for (2.7b), we have

$$\beta_m C^i_{jkh} = \beta_m W^i_{jkh} - \tfrac{5}{6}\beta_m(\delta^i_k R_{jh} - R^i_h g_{jk}) - \tfrac{1}{6}\beta_m R(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$
$$+\tfrac{1}{2}\beta_m(\delta^i_h R_{jk} - R^i_k g_{jh}) \ . \tag{4.10}$$

From (1.7a), (1.7b), (1.7d), (4.1) and (4.10), we get

$$\beta_m C^i_{jkh} = \lambda_m \left(W^i_{jkh} - \tfrac{5}{6}(\delta^i_k R_{jh} - R^i_h g_{jk}) - \tfrac{1}{6}R(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{2}(\delta^i_h R_{jk} - R^i_k g_{jh})\right)$$
$$+ \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \tfrac{1}{4}(R^h_k g_{ij} - R^h_j g_{ik}) \ . \tag{4.11}$$

By using (2.7b) in (4.11), we have

$$\beta_m C^i_{jkh} = \lambda_m C^i_{jkh} + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \tfrac{1}{4}(R^h_k g_{ij} - R^h_j g_{ik}) \ . \tag{4.12}$$

In conclusion the proof of theorem is completed, we can determine

Theorem 4.4. The expansion derivative for Berwald of Conformal curvature tensor C^h_{ijk} in (2.7b) is satisfies the equation (4.12).

4-4. Tack away Berwald covariant derivative for (2.8b), we have

$$\beta_m L^i_{jkh} = \beta_m W^i_{jkh} + \tfrac{1}{6}\beta_m(\delta^i_k R_{jh} - R^i_h g_{jk}) - \tfrac{1}{2}\beta_m(\delta^i_h R_{jk} - R^i_k g_{jh}) \ . \tag{4.13}$$

From (1.7a), (1.7b), (4.1) and (4.13), we get

$$\beta_m L^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh})$$
$$+\tfrac{1}{6}\lambda_m(\delta^i_k R_{jh} - R^i_h g_{jk}) - \tfrac{1}{2}\lambda_m(\delta^i_h R_{jk} - R^i_k g_{jh}) \ .$$

Or can be written as

$$\beta_m L^i_{jkh} = \lambda_m \left(W^i_{jkh} + \tfrac{1}{6}(\delta^i_k R_{jh} - R^i_h g_{jk}) - \tfrac{1}{2}(\delta^i_h R_{jk} - R^i_k g_{jh})\right)$$
$$+ \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) \ . \tag{4.14}$$

From (2.8b) and (4.14), we get

$$\beta_m L^i_{jkh} = \lambda_m \ L^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) \ . \tag{4.15}$$

Thus, the proof of theorem is completed, we get

Theorem 4.5. The expansion derivative for Berwald of Conharmonic curvature tensor L^i_{jkh} in (2.8b) is satisfies the equation (4.15).

4-5. Tack away Berwald covariant derivative for (2.10), we have

$$\beta_m M^i_{jkh} = \beta_m W^i_{jkh} - \tfrac{1}{12}\beta_m R(g_{jk}\delta^i_h - g_{jh}\delta^i_k) - \tfrac{1}{6}\beta_m(\delta^i_k R_{jh} - R^i_h g_{jk}) \ . \tag{4.16}$$

From (1.7a), (1.7b), (1.7d), (4.1) and (4.16), we get

$$\beta_m M^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh})$$
$$-\tfrac{1}{12}\lambda_m R(g_{jk}\delta^i_h - g_{jh}\delta^i_k) - \tfrac{1}{6}\lambda_m(\delta^i_k R_{jh} - R^i_h g_{jk}) \ .$$

Or can be written as

$$\beta_m M^i_{jkh} = \lambda_m \left(W^i_{jkh} - \tfrac{1}{12}R(g_{jk}\delta^i_h - g_{jh}\delta^i_k) - \tfrac{1}{6}(\delta^i_k R_{jh} - R^i_h g_{jk})\right)$$

$$+\mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) \ . \tag{4.17}$$

From (2.10) and (4.17), we have

$$\beta_m M^i_{jkh} = \lambda_m M^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}] \,. \tag{4.18}$$

In conclusion the proof of theorem is completed, we can determine

Theorem 5.6. The expansion derivative for Berwald of Concircular curvature tensor M^i_{jkh}in (2.10) is satisfies the equation (4.18).

4-6. Tack away Berwald covariant derivative for (2.14), we have

$$\beta_m P^i_{1_{jkh}} = \beta_m W^i_{jkh} + \tfrac{1}{6}\beta_m[\delta^i_h R_{jk} - g_{jh} R^i_k] - \tfrac{1}{3}\beta_m[\ \delta^i_k R_{jh} - g_{jk} R^i_h] \,. \tag{4.19}$$

From (1.7a), (1.7b), (4.1) and (4.19), we get

$$\beta_m P^i_{1_{jkh}} = \lambda_m \left(W^i_{jkh} + \tfrac{1}{6}\ (\delta^i_h R_{jk} - g_{jh} R^i_k) - \tfrac{1}{3}(\delta^i_k R_{jh} - g_{jk} R^i_h)\right)$$

$$+\ \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) \,. \tag{4.20}$$

By using (2.14) in (4.20), we have

$$\beta_m P^i_{1_{jkh}} = \lambda_m P^i_{1_{jkh}} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(W^i_h g_{jk} - W^i_k g_{jh}) \ \ . \tag{4.21}$$

The proof of theorem is completed, we conclude

Theorem 5.7. The expansion derivative for Berwald of P1-curvature tensor $P^i_{1_{jkh}}$ in (2.14) is satisfies the equation (4.21).

Transvecting condition to a higher dimensional space (4.1) by y^j, using (1.2b), (2.3a) and (1.4b), we get

$$\beta_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{4}(W^i_h y_k - W^i_k y_h) \ . \tag{4.22}$$

Again, transvecting condition to a higher dimensional space (4.22) by y^k, using (1.2b), (2.3b), (2.3c), (1,11a) and (1.11b), we get

$$\beta_m W^i_h = \lambda_m W^i_h + \mu_m(y^i y_k - \delta^i_k F^2) + \tfrac{1}{4} W^i_h F^2 \,. \tag{4.23}$$

Therefore, the proof of theorem is completed, we can say

Theorem 5.8. In covariant derivative for Berwald of first order for torsion tensor W^i_{kh} and deviation tensor W^i_h are given by (4.22) and (4.23).

Contracting the indices i and h in the equations (4.5) and (4.6), respectively and using (1.4a), (1.4b), (1.8a), (1.5c), (1.5d), and (2.3d), we get

$$\beta_m H_k = \lambda_m \mathrm{H}_k + \mu_m(n-1)\ y_k - \tfrac{1}{4} W^i_k y_i \ . \tag{4.24}$$

And

$$\beta_m H = \lambda_m H + \mu_m(n-1)\ F^2 \,. \tag{4.25}$$

Thus, we conclude

Theorem 5.9. In covariant derivative for Berwald of firest order for vector H_k and scalar H are given by (4.24) and (4.25).

6. Conclusion

In this work, we have investigated the expansion properties of several curvature tensors in Finsler spaces with respect to the Berwald covariant derivative. Starting from the Weyl projective curvature tensor, various curvature tensors including the Riemannian, projective, conformal, conharmonic, concircular, and P_1-curvature tensors were expressed in a unified form and analyzed within the framework of Finsler geometry. A generalized expansion curvature tensor was introduced, and explicit expansion formulas were derived. By establishing a series of identities, we proved several theorems describing the behavior of these curvature tensors under the Berwald covariant derivative. The obtained results demonstrate that the expansion properties of different curvature tensors follow a common structural pattern, highlighting deep interrelations among them in Finsler spaces. The results presented in this paper extend well-known concepts from Riemannian geometry to the more general setting of Finsler geometry. They provide a useful foundation for further studies on curvature structures, geometric invariants, and their potential applications in mathematical physics and generalized theories of spacetime.

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توسيع موترات الانحناء بواسطة المشتقة التغايرية لبرفالد في فضاءات فنسلر

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الملخص: تُعد موترات الانحناء من الأدوات الأساسية في الهندسة التفاضلية لوصف البنية الهندسية للمتشعبات. في هذا البحث، ندرس توسّع عدد من موترات الانحناء في فضاءات فنسلر باستخدام المشتقة التغايرية لبرفالد. حيث يتم التعبير عن عدة موترات انحناء، من بينها موتر الانحناء الريماني، والموتر الإسقاطي، والموتر المطابقي، وموتر الانحناء الكونهرموني، وموتر الانحناء الدائري، وموتر الانحناء P_1، بدلالة موتر الانحناء الإسقاطي لـويل.

كما تم اشتقاق صيغة توسّع عامة، وإثبات عدد من الهويات والنظريات المتعلقة بالمشتقات التغايرية لبرفالد لهذه الموترات. وتُعد النتائج المتحصَّل عليها تعميمًا للعلاقات الكلاسيكية في الهندسة الريمانية إلى إطار هندسة فنسلر، وتسهم في دراسة بُنى الانحناء في المتشعبات الفنسلرية.

الكلمات المفتاحية: موتر الانحناء الريماني؛ المشتقة التغايرية لبرفالد؛ موتر الانحناء الإسقاطي لـويل؛ موتر الانحناء المطابقي؛ موتر الانحناء الكونهرموني؛ موتر الانحناء الدائري.