# Geometric Properties and Recurrence Conditions in Generalized $H^v$-Recurrent Finsler Spaces with Applications to Differential Geometric Modelling

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**Abstract:** This paper investigates the geometric structure and recurrence properties of generalized $H^v$-recurrent Finsler spaces $GH^v$-$RF_n$, characterized by a Berwald curvature tensor $H^i_{jkh}$ satisfying a specific (v)-covariant recurrence condition. Several fundamental identities are derived for the first-order (v)-covariant derivatives of the Berwald curvature tensor, the deviation tensor, the Ricci tensor, the curvature vector, and the associated torsion and curvature tensors. Necessary and sufficient conditions for these tensors to be generalized recurrent are established through detailed analysis using Cartan's first and second kind covariant differential operators. The results show that the Berwald Ricci tensor and the Berwald scalar curvature cannot vanish in such spaces, and that several geometric objects exhibit strict recurrence behavior governed by the recurrence vectors $\lambda_l$ and $\mu_l$. Furthermore, generalized recurrence conditions for Izumi's tensor $Z^i_{jkh}$ are obtained, along with new identities linking various torsion and curvature components.

The theoretical framework developed in this study provides deeper insight into the structure of Finsler spaces with generalized recurrence and supports potential applications in engineering fields that rely on differential geometric modelling, such as nonlinear mechanical systems, anisotropic material modelling, and advanced control systems.

**Keywords:** Finsler geometry, Generalized $H^v$-recurrent spaces, Berwald curvature, Recurrence tensors, Differential geometric modeling, Covariant differentiation, Izumi tensor.

**1. Introduction:** Finsler geometry provides a rich generalization of Riemannian geometry through the introduction of directional dependence in the metric structure, leading to a wide spectrum of curvature and torsion tensors with intricate geometric behavior. Among these tensors, the Berwald curvature and its associated torsion and deviation tensors play a fundamental role in characterizing the geometric and physical properties of Finsler spaces. Recurrent structures in differential geometry, particularly within the Finslerian framework, have attracted considerable attention due to their relevance in the study of symmetric spaces, stability of geodesic flows, and applications in engineering models involving anisotropic or nonlinear media.

In this context, generalized recurrent spaces constitute a natural extension of classical recurrent manifolds, where curvature-related tensors satisfy specific recurrence relations governed by non-vanishing covariant vector fields. Recently, the notion of generalized $H^h$-recurrent Finsler spaces has been introduced, providing a coherent framework for analyzing recurrence phenomena through the (h)-covariant derivative. Motivated by this development, the present work focuses on the complementary structure governed by the (v)-covariant derivative and introduces the class of generalized $H^v$-recurrent Finsler spaces, denoted by $GH^v$-$RF_n$. These spaces are characterized by a Berwald curvature tensor $H^i_{jkh}$ satisfying a specific (v)-recurrence condition involving two non-null recurrence vectors $\lambda_l$ and $\mu_l$.

The objective of this paper is to investigate the geometric properties of such spaces and to establish a set of fundamental theorems describing the behavior of various curvature and torsion tensors under (v)-covariant differentiation. By employing Cartan's first and second kind covariant differential operators, we obtain explicit expressions for the first-order (v)-covariant derivatives of the Berwald curvature tensor, the deviation tensor, the torsion tensor, the Ricci tensor, the curvature scalar, and the associate curvature tensor. Furthermore, the study demonstrates that several of these geometric objects are necessarily non-vanishing in generalized $H^v$-recurrent spaces, and that their recurrence behavior is tightly controlled by the recurrence vectors.

In addition, the paper establishes necessary and sufficient conditions for some of these tensors to be generalized recurrent, particularly through detailed identities involving the Cartan tensor and its derivatives. Special attention is given to Izumi's tensor $Z^i_{jkh}$, for which a generalized recurrence condition is derived based on the recurrence properties of an associated mixed tensor. The obtained results culminate in a set of identities that characterize the geometric structure of $GH^v$-$RF_n$ spaces and enrich the theory of recurrent Finsler manifolds.

The findings presented in this work deepen the understanding of recurrence phenomena in Finsler geometry and contribute to differential geometric modeling relevant to modern engineering applications, especially those involving anisotropic materials, nonlinear dynamical systems, and geometric control theory.

Previous research on curvature tensors and recurrent structures in Finsler geometry has established a substantial foundation for the present study. Early contributions, such as those by Dubey and Srivastava (1981), De and Guha (1991), and Matsumoto (1971), laid the groundwork for understanding recurrent and generalized recurrent manifolds within the broader context of differential geometry. Subsequent developments expanded these concepts to higher-order recurrences and specialized curvature tensors, as seen in the works of Dikshit (1992), Mishra and Lodhi (2008), and Misra et al. (2014). In recent years, a significant body of literature has focused on advanced curvature structures, higher-order derivatives, and decomposition techniques in Finsler spaces. Ahsan and Ali (2014, 2016) examined fundamental properties of various curvature tensors, while numerous studies by Al-Qashbari and collaborators (2017–2025) introduced extensive generalizations of recurrent Finsler structures, including BR-trirecurrent, $W^h$-birecurrent, and higher-order recurrent spaces. Their works also explored Berwald and Cartan covariant derivatives, Lie derivatives, and specialized tensors such as Weyl, M-projective, R-projective, and conharmonic curvature tensors. These contributions have enriched the analytical framework of Finsler geometry and provided new methods for characterizing generalized recurrent manifolds.

Collectively, the existing literature demonstrates the depth and evolution of research on curvature tensors and recurrence phenomena in Finsler geometry. The present work builds upon these advances by further developing the structural understanding of generalized recurrent spaces and extending the theoretical tools used to analyze higher-order curvature behaviors.

Let us consider an n-dimensional Finsler space equipped with the metric function F satisfying the requisite conditions. Let consider the components of the corresponding metric tensor $g_{ij}$, Cartan's connection parameters $\Gamma^{*i}_{jk}$ and Berwald's connection parameters $G^i_{jk}$. These are symmetric in their lower indices and positively homogeneous of degree zero in the directional arguments.

The two sets of quantities $g_{ij}$ and its associate tensor $g^{ij}$ are related by

(1.1) $$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1\ , & if \quad i = k \quad , \\ 0\ , & if \quad i \neq k \quad . \end{cases}$$

The vectors $y_i$ and $y^i$ satisfies the following relations

(1.2) a) $y_i = g_{ij}\, y^j$ , b) $y_i\, y^i = F^2$ , c) $g_{ij} = \dot{\partial}_i\, y_j = \dot{\partial}_j\, y_i$ ,

d) $g_{ij}\, y^j = \frac{1}{2}\dot{\partial}_i F^2 = F\dot{\partial}_i F$ and e) $\dot{\partial}_j\, y^i = \delta^i_j$ .

The tensor $C_{ijk}$ defined by

(1.3) $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\,\dot{\partial}_j\,\dot{\partial}_k\, F^2$

is known as (h) hv - torsion tensor. It is positively homogeneous of degree -1 in the directional arguments and symmetric in all its indices.

The (v) hv-torsion tensor $C^h_{ik}$ and its associate (h) hv-torsion tensor $C_{ijk}$ are related by

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ , b) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$

c) $C^h_{ik} = g^{hj}\, C_{ijk}$ and d) $\left(\dot{\partial}_j\, C^i_{hk}\right) y^k = -\, C^i_{hj}$ .

The (v) hv-torsion tensor $C^h_{ik}$ is also positively homogeneous of degree -1 in the directional arguments and symmetric in its lower indices.

Eliminating $dy^k$ from (1.5.1) and in terms of the absolute differential of $l^i$ , É. Cartan deduced

(1.5) $DX^i = FX^i|_k\, Dl^k + X^i_{|k} dx^k + y^k\left(\dot{\partial}_k X^i\right)\frac{dF}{F}$ , where $X^i|_k = \dot{\partial}_k X^i + X^r C^i_{rk}$ .

The unit vector $l^i$ and the metric function F are vanished under v-covariant differentiation, i.e.

(1.6) a) $l^i|_k = 0$ and b) $F|_k = 0$ .

The v-covariant derivative of metric tensors $g_{ij}$ , $g^{ij}$ and the vectors $y^i$ , $y_i$ are given by

(1.7) a) $g_{ij}|_k = 0$ , b) $g^{ij}|_k = 0$ , c) $y_i|_k = g_{ik}$ and d) $y^i|_k = \delta^i_k$ .

And the metric tensor $g_{ij}$ and the vector $y_i$ are given by

(1.8) a) $g_{ij}\delta^i_k = g_{jk}$ and b) $y_i\delta^i_k = y_k$ .

For an arbitrary vector field $x^i$ , the two Processes of covariant differentiation, defined above commute with partial differentiation with respect to $y^j$ according to

(1.9) $\dot{\partial}_j(X^i|_k) - \left(\dot{\partial}_j\, X^i\right)|_k = X^h\left(\dot{\partial}_j C^i_{kh}\right) + C^h_{kj}\left(\dot{\partial}_h X^i\right)$ .

The quantities $H^i_{jkh}$ and $H^i_{kh}$ form the components of tensors and they called h-curvature tensor of Berwald (Berwald curvature tensor) and h(v)-torsion tensor, respectively, and defined as follow:

(1.10) a) $H^i_{jkh} = \partial_j\, G^i_{kh} + G^r_{kh} G^i_{rj} + G^i_{rhj}\, G^r_k - \partial_j\, G^i_{hk} - G^r_{hk}\, G^i_{rj} - G^i_{rkj}\, G^r_h$ ,

An b) $H^i_{kh} = \partial_h\, G^i_k + G^r_k\, C^i_{rh} - \partial_k\, G^i_h - G^r_h\, C^i_{rk}$ .

They are skew-symmetric in their lower indices, i.e. $k$ and $h$ . Also, they are positively homogeneous of degree zero and one, respectively in their directional arguments.

Contraction of the indices $i$ and $h$ in (1.10a) and (1.10b), we get the following

(1.11) a) $H^i_{jki} = H_{jk}$ and b) $H^i_{ki} = H_k$ .

They are also related by

(1.12) a) $H^i_{jkh} y^j = H^i_{kh}$ , b) $H^i_{jkh} = \partial_j H^i_{kh}$ , c) $H^i_{jk} = \partial_j H^i_k$ and d) $H_{jk} = \partial_j H_k$ .

These tensors were constructed initially by mean of the tensor $H^i_h$ , called the deviation tensor, given by

(1.13) $H^i_h = 2\partial_h G^i - \partial_r G^i_h\, y^r + 2G^i_{hs} G^s - G^i_s G^s_h$ .

The deviation tensor $H^i_h$ is positively homogeneous of degree two in the directional arguments.

The h(v)-torsion tensor $H^i_{kh}$ , the deviation tensor $H^i_k$, the curvature vector $H_k$ and the scalar curvature $H$ in sense of Berwald, is given by

(1.14) a) $H^i_{jk}\, y^j = -\, H^i_{kj}\, y^j = H^i_k$ , b) $y_i\, H^i_j = 0$ , c) $H_i y^i = (n-1)\, H$ ,

d) $H_{ji} y^i = (n-1)\,\dot{\partial}_j H - H_j$ and e) $H_{kh}\, y^h = (n-1)\,\dot{\partial}_k\, H - H_k$ .

The quantities $H^i_{jkh}$ and $H^i_{kh}$ are satisfies the following

(1.15) a) $H_{ijkh} = g_{jr} H^r_{ihk}$ , b) $H_{jk.h} = g_{jr} H^r_{hk}$ and c) $H^i_{ikh} = (H_{hk} - H_{kh})$ .

Also, Berwald curvature tensor $H^i_{jkh}$ satisfies the following:

(1.16) a) $\left(\dot{\partial}_j H^i_{kh}\right) y^k = \left(\dot{\partial}_j H^i_{kh}\right) y^h = 0$ , and b) $\left(H^i_{jkh}|_m\right) y^j = \left(H^i_{jkh}\, y^j\right)|_m - H^i_{jkh}\delta^j_m$ .

## 2. Properties of Generalized $H^v$-Recurrent Finsler Spaces

This section develops the fundamental geometric properties of generalized $H^v$-recurrent Finsler spaces, denoted by $GH^v$-$RF_n$. These spaces are characterized by a Berwald curvature tensor whose (v)-covariant derivative satisfies a specific recurrence condition governed by two non-null vector fields. Using Cartan's first kind covariant differential operator, several identities are derived for the (v)-covariant derivatives of the Berwald curvature tensor and its associated geometric objects. The section establishes essential theorems concerning torsion behavior, deviation tensors, Ricci-type tensors, curvature vectors, and associated curvature structures.

Let $F_n$ be a Finsler space with Berwald curvature tensor $H^i_{jkh}$ . A space is said to be generalized $H^v$-recurrent if the Berwald curvature satisfies the relation

(2.1) $$H^i_{jkh}|_l = \lambda_l\, H^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \quad , \quad H^i_{jkh} \neq 0 \quad ,$$

where $|_l$ denotes the (v)-covariant derivative, and $\lambda_l$ , $\mu_l$ are non-zero recurrence vectors. Such a space is denoted by $GH^v$-$RF_n$.

**Definition 2.1.** A Finsler space $F_n$ whose Berwald curvature tensor $H^i_{jkh}$ is satisfying the condition 2.1, where $\lambda_l$ and $\mu_l$ are non-null covariant vectors field, is called a generalized $H^v$-recurrent space and the tensor will be called generalized v-recurrent tensor, respectively. We shall denote them briefly by $GH^v$-$RF_n$ and $Gv$-$R$ , respectively.

Transvecting the above condition by $y^j$ and employing standard identities of Finsler geometry (1.11a), (1.6d), (1.1), (1.2b) and (1.15b) yields

(2.2) $$H^i_{kh}|_l = \lambda_l H^i_{kh} + H^i_{lkh} + \mu_l(\delta^i_h\, y_k - \delta^i_k\, y_h) \quad .$$

This leads to the following result.

**Theorem 2.1.** In a generalized $H^v$-recurrent Finsler space $GH^v$-$RF_n$, the first-order (v)-covariant derivative of the torsion tensor $H^i_{kh}$ satisfies the relation above (2.2).

Further, transvecting (2.2) by $y^k$ , using (1.6d), (1.1), (1.2b), (1.15b), (1.13a) and (1.15a), we get

(2.3) $$H^i_h|_l = \lambda_l H^i_h + \mu_l(\delta^i_h\, F^2 - y_h\, y^i) \quad .$$

leading to:

**Theorem 2.2.** In a $GH^v$-$RF_n$ space, the deviation tensor $H^i_h$ satisfies the foregoing recurrence relation under (v)-covariant differentiation.

Contracting indices appropriately yields recurrence relations for the Berwald Ricci tensor and the curvature scalar:

(2.4) $$H_{jk}|_l = \lambda_l H_{jk} + (n-1)\, \mu_l g_{jk} \quad , \text{ and}$$

(2.5) $$H|_l = \lambda_l H + \mu_l\, F^2 \quad .$$

The conditions (2.4) and (2.5), show that Ricci tensor $H_{jk}$ and the curvature scalar $H$ (both in sense of Bewald) cannot vanish, since the vanishing of any one of them would imply the vanishing of the covariant vector field $\mu_l$ , i.e. $\mu_l = 0$ , a contradiction.

Since vanishing of either quantity forces $\mu_l = 0$, a contradiction, we deduce:

**Theorem 2.3.** In a $GH^v$-$RF_n$ space, both the Berwald Ricci tensor $H_{jk}$ and the Berwald scalar curvature $H$ are necessarily non-zero.

Similar contraction gives

(2.6) $$H_k|_l = \lambda_l H_k + H_{lk} + (n-1)\, \mu_l\, y_k \quad .$$

which establishes:

**Theorem 2.4.** The curvature vector $H_k$ is non-vanishing in any $GH^v$-$RF_n$.

By transvecting with $g_{ip}$ , additional equivalent forms of the recurrence condition are obtained:

(2.7) $H_{jpkh}|_l = \lambda_l \, H_{jpkh} + \mu_l(g_{hp}\, g_{jk} - g_{kp}\, g_{jh})$ , and

(2.8) $H_{kp.h}|_l = \lambda_l \, H_{kp.h} + H_{lpkh} + \mu_l(g_{hp}\, y_k - g_{kp}\, y_h)$ ,

Conversely, transvecting conditions (2.7) and (2.8) by $g^{ip}$ yields conditions (2.1) and (2.2), respectively. Consequently, conditions (2.1) and (2.2) are equivalent to conditions (2.7) and (2.8), respectively. Hence, a generalized $H^v$-recurrent space can be fully characterized by condition (2.7), while the first-order $v$-covariant derivative of the associated torsion tensor $H_{kp,h}$ associated with the $h(v)$-torsion tensor $H^i_{kh}$ is given by condition (2.8).

We thus obtain the following results:

**Theorem 2.5.** A generalized $GH^v$-$RF_n$ space is characterized by condition (2.7).

**Theorem 2.6.** In $GH^v$-$RF_n$ , space, the first-order $v$-covariant derivative of the associated torsion tensor $H_{kp,h}$ associated with the $h(v)$-torsion tensor $H^i_{kh}$ is determined by condition (2.8).

Contracting the indices $i$ and $j$ in (2.1), and using (1.15c) and (1.1), we obtain:

$$(H_{hk} - H_{kh})|_l = \lambda_l (H_{hk} - H_{kh}) \quad .$$

Thus, we conclude

**Theorem 2.7.** In $GH^v$-$RF_n$ , the curvature tensor $(H_{hk} - H_{kh})$ is recurrent.

Transvecting the conditions (2.4) and (2.6) by $y^k$, using (1.6d), (1.14c), (1.2a) and (1.14d), we get

$$[(n-1)\,\dot\partial_j H - H_j]|_l = \lambda_l[(n-1)\,\dot\partial_j H - H_j] + H_{jl} + (n-1)\,\mu_l\, y_j \quad , \text{ and}$$

$$H|_l = \lambda_l\, H + \dot\partial_j\, H + \mu_l\, F^2 \quad , \text{respectively.}$$

Thus, we conclude

**Theorem 2.8.** In $GH^v$-$RF_n$ , the tensor $[(n-1)\,\dot\partial_j H - H_j\,]$ and the curvature scalar $H$ (in sense of Bewald) are non-vanishing if and only if $H_{jl} = 0$ and $\dot\partial_j H = 0$ , respectively.

Now, we know that Berwald curvature tensor $H^i_{jkh}$ and the h(v)-torsion tensor $H^i_{jk}$ satisfy the equation

(2.9) $g_{ip}\, H^i_{jkh} + g_{ij}\, H^i_{pkh} = 0$ .

Taking the v-covariant derivative for the above equation with respect to $y^l$, using (1.2d), we get

(2.10) $g_{il}\, H^i_{jkh} + y_i\, H^i_{jkh}|_l + g_{ih}\, H^i_{jk}|_l = 0$ .

By employing (1.15a), and conditions (1.7a) and (1.7b) in (2.10), we get

(2.11) $H_{jlkh} + y_i[\lambda_l H^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})] + g_{ih}[\lambda_l H^i_{jk} + H^i_{ljk} + \mu_l(\delta^i_k\, y_j - \delta^i_j\, y_k)] = 0$ .

Using (1.15b) and (1.7a), equation (2.11) becomes:

(2.12) $\lambda_l(y_i\, H^i_{jkh} + g_{ih}\, H^i_{jk}) + H_{jlkh} + H_{lhjk} + \mu_l(g_{jk} y_h + g_{kh}\, y_j - 2 g_{jh}\, y_k) = 0$ .

Substituting the equation (2.9) in (2.12), we get

(2.13) $H_{jlkh} + H_{lhjk} + \mu_l(g_{jk}\, y_h + g_{kh}\, y_j - 2\, g_{jh} y_k) = 0$ .

Transvecting (2.13) by $y^k$ and $y^h$ separately, using (1.2a) and (1.2b), we get

(2.14) $H_{jlkh}\, y^k y^h = -\, H_{lhjk}\, y^k y^h$ .

Therefore, we have the identity for the associate curvature tensor $H_{jlkh}$ of Berwald curvature tensor $H^i_{jkh}$ in a generalized $H^v$- recurrent space $F_n$ given by the condition (2.14).

Thus, we conclude

**Theorem 2.9.** In $GH^v$-$RF_n$ , we have the identity (2.14) for the associate curvature tensor $H_{jlkh}$ of Berwald curvature tensor $H^i_{jkh}$ .

## 3. Necessary and Sufficient Conditions for Generalized Recurrence of Selected Tensors in $GH^v$-$RF_n$

In this section, we derive the necessary and sufficient conditions under which several geometric tensors in a generalized $H^v$-recurrent Finsler space $GH^v$-$RF_n$become generalized recurrent. By applying partial differentiation with respect to directional arguments and utilizing the commutation formulas associated with the Berwald connection, we obtain recurrence criteria for the Berwald curvature tensor, the Ricci-type tensor, and the Izumi tensor $Z^i_{jkh}$. These results culminate in a set of structural identities that characterize generalized recurrence within this geometric framework.

Differentiating condition (2.3) partially with respect to $y^j$ , using (1.12b) and (1.2c), we get

(3.1) $$\dot\partial_j(H^i_{kh}|_l) = (\dot\partial_j\lambda_l)\, H^i_{kh} + \lambda_l(H^i_{jkh}) + \dot\partial_j H^i_{lk\square}$$
$$+ (\dot\partial_j\, \mu_l)(\delta^i_h\, y_k - \delta^i_k\, y_h) + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \quad .$$

Using the commutation formula exhibited by (1.8) for the h(v) torsion tensor $H^i_{jk}$ in the condition (3.1) and using (1.12b), we get

(3.2) $$H^i_{jkh}|_l + H^r_{kh}(\dot\partial_j C^i_{lr}) - H^i_{rh}(\dot\partial_j C^r_{lk}) - H^i_{kr}(\dot\partial_j C^r_{lh}) + C^r_{lj}H^i_{rkh} = (\dot\partial_j\lambda_l)\, H^i_{kh} + \lambda_l\, H^i_{jkh}$$
$$+\dot\partial_j\, H^i_{lkh} + (\dot\partial_j\, \mu_l)(\delta^i_h\, y_k - \delta^i_k\, y_h) + \mu_l\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \;.$$

This shows that

$$H^i_{jkh}|_l = \lambda_l H^i_{jkh} + \mu_l\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$$

if and only if

(3.3) $$H^r_{kh}(\dot\partial_j C^i_{lr}) - H^i_{rh}(\dot\partial_j C^r_{lk}) - H^i_{kr}(\dot\partial_j\, C^r_{lh}) + C^r_{lj}\, H^i_{rkh}$$
$$= (\dot\partial_j\, \lambda_l)\, H^i_{kh} + \dot\partial_j\, H^i_{lkh} + (\dot\partial_j\, \mu_l)(\delta^i_h\, y_k - \delta^i_k\, y_h) \quad .$$

Thus, we conclude

**Theorem 3.1.** In $GH^v$-$RF_n$, the Berwald curvature tensor $H^i_{jkh}$is generalized v-recurrent if and only if condition (3.3) holds.

Differentiating the equation (2.6) partially with respect to $y^j$ , using (1.12c) and (1.2c), we get

(3.4) $$\dot\partial_j(H_k|_l) = (\dot\partial_j\lambda_l)\, H_k + \lambda_l(H_{jk}) + \dot\partial_j\, H_{lk} + (n-1)\,(\dot\partial_j\, \mu_l)\, y_k + (n-1)\, \mu_l\, g_{jk} \;.$$

Using the commutation formula exhibited by (1.8) for the curvature vector $H_k$ in (3.4) and using (1.12d), we get

(3.5) $$H_{jk}|_l - H_r(\dot\partial_j\, C^r_{kl}) + H_{rk}\, C^r_{jl} = (\dot\partial_j\, \lambda_l)\, H_k + \lambda_l H_{jk} + \dot\partial_j H_{lk}$$
$$+ (n-1)\,(\dot\partial_j\, \mu_l)\, y_k + (n-1)\, \mu_l g_{jk} \quad .$$

This shows that

$$H_{jk}|_l = \lambda_l\, H_{jk} + (n-1)\, \mu_l\, g_{jk}$$

if and only if

(3.6) $$H_{rk}C^r_{jl} - H_r(\dot\partial_j C^r_{kl}) = (\dot\partial_j\lambda_l)H_k + \dot\partial_j H_{lk} + (n-1)(\dot\partial_j\, \mu_l)\, y_k \quad .$$

Thus, we conclude

**Theorem 3.2.** In $GH^v$-$RF_n$ , Ricci tensor $H_{jk}$ (in sense of Berwald) is non-vanishing if and only if (3.6) holds.

H. Izumi [20] defined the tensor $Z^i_{jkh}$ as the follows:

(3.7) $$Z^i_{jkh} = H^i_{jkh} - \frac{1}{(n-1)}\left(H^{*i}_h g_{jk} - H^{*i}_k g_{jh}\right) \text{, where } H^i_{jkh}\, g^{jk} = H^{*i}_h \;.$$

Taking the v-covariant derivative for the condition (3.7) with respect to $y^l$, we get

(3.8) $$Z^i_{jkh}|_l = H^i_{jkh}|_l - \frac{1}{(n-1)}\left(H^{*i}_h g_{jk} - H^{*i}_k g_{jh}\right)|_l \;.$$

Using the condition (2.1) in (3.8), we get

$$Z^i_{jkh}|_l = \lambda_l H^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) - \frac{1}{(n-1)}\left(H^{*i}_h g_{jk} - H^{*i}_k g_{jh}\right)|_l \;.$$

In view of the condition (3.7), the above equation becomes

$$Z^i_{jkh}|_l = \lambda_l Z^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) + \frac{1}{(n-1)}\lambda_l\left(H^{*i}_h g_{jk} - H^{*i}_k g_{jh}\right)$$
$$- \frac{1}{(n-1)}\left(H^{*i}_h g_{jk} - H^{*i}_k g_{jh}\right)|_l \;.$$

This shows that

$$Z^i_{jkh}|_l = \lambda_l\, Z^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$$

if and only if

$$\left(H^{*i}_h g_{jk} - H^{*i}_k g_{jh}\right)|_l = \lambda_l\left(H^{*i}_h g_{jk} - H^{*i}_k g_{jh}\right) \;.$$

Thus, we conclude

**Theorem 3.3.** In $GH^v$-$RF_n$ , the tensor $Z^i_{jkh}$ is generalized recurrent if and only if the tensor $\left(H^{*i}_h g_{jk} - H^{*i}_k g_{jh}\right)$ is recurrent.

Using (2.1) and (2.6) in (3.2) and (3.5), we get

(3.9) $$H^r_{kh}(\dot\partial_j\, C^i_{lr}) - H^i_{rh}\,(\dot\partial_j\, C^r_{lk}) - H^i_{kr}(\dot\partial_j C^r_{lh}) + C^r_{lj}\, H^i_{rkh}$$
$$= (\dot\partial_j\, \lambda_l)\, H^i_{kh} + \dot\partial_j\, H^i_{lkh} + (\dot\partial_j\, \mu_l)(\delta^i_h\, y_k - \delta^i_k\, y_h) \;,$$

and

(3.10) $$H_{rk} C^r_{jl} - H_r(\dot\partial_j C^r_{kl}) = (\dot\partial_j \lambda_l)\, H_k + \dot\partial_j H_{lk} + (n-1)(\dot\partial_j\, \mu_l)\, y_k \;,$$

respectively.

Transvecting (3.10) by $y^k$, using (1.14a), (1.4d), (1.16a), (1.7a) and (1.2b), we get

(3.11) $$H^r_h(\dot\partial_j C^i_{lr}) + H^i_{rh} C^r_{lj} - H^i_r(\dot\partial_j C^r_{lh}) = (\dot\partial_j\, \lambda_l)\, H^i_h + (\dot\partial_j\, \mu_l)(\delta^i_h\, F^2 - y_h\, y^i) \;.$$

Further, transvection (3.10) by $y^k$, using (1.14d), (1.4d), (1.14c), (1.8.18a) and (1.7a), we get

(6.3.12) $$(n-1)(\partial_r H)\, C^r_{jl} = (n-1)(\dot\partial_j\, \lambda_l)\, H + (n-1)\, \dot\partial_j\, \partial_l\, H - H_{lj}$$
$$- H_{jl} + (n-1)(\dot\partial_j\, \mu_l)\, F^2 \;.$$

Thus, we conclude

**Theorem 3.4.** In $GH^v$-$RF_n$ , identities (3.9), (3.10), (3.11) and (312) hold.

## 4. Applications of Generalized $H^v$-Recurrent Finsler Structures

The recurrence relations established for the Berwald curvature and the associated geometric tensors in $GH^{v}\text{-}RF_n$ spaces provide an effective framework for modelling a wide range of anisotropic geometric phenomena. Owing to the presence of two non-null recurrence vectors $\lambda_l$ and $\mu_l$ , these structures naturally capture direction-dependent behavior in curvature, torsion, and Ricci-type quantities.

### 4.1. Modelling Anisotropic Curvature Fields

The fundamental recurrence condition: $H^i_{jkh}|_l = \lambda_l\, H^i_{jkh} + \mu_l\big(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\big)$ ,

allows describing the evolution of curvature along distinguished directions.

This is directly applicable to:

a) anisotropic optical and material media,
b) geometric stability analyses of Finslerian trajectories,
c) and curvature-driven propagation in direction-dependent systems.

The guaranteed non-vanishing of the Ricci tensor, curvature scalar, and curvature vector (Theorems 2.3–2.4) ensures persistent geometric anisotropy in such models.

### 4.2. Recurrence-Based Tensor Modelling

The necessary and sufficient conditions obtained in (3.3), (3.6), and Theorems 3.1-3.3 provide criteria under which the Berwald curvature, the Ricci-type tensor, and the Izumi tensor become generalized recurrent. These results support:

a) Finsler Ricci flow modelling under structured tensor evolution,
b) anisotropic deformation analysis in geometric mechanics and biological tissues,
c) and invariant field construction through recurrent curvature objects.

### 4.3. Associated Curvature and Structural Invariants

The equivalence of conditions (2.1), (2.7), and (2.8) (Theorems 2.5-2.6) enables characterizing $GH^{v}\text{-}RF_n$ spaces through associated curvature tensors. The identity (2.14) (Theorem 2.9) further imposes symmetry constraints that are useful in:

a) classification of anisotropic geometric structures,
b) modelling higher-order curvature interactions,
c) and analyzing wave-front deformation in Finsler-type optical geometries.

### 4.4. Scalar Curvature Behaviour in Differential Models

The scalar recurrence relation: $H|_l = \lambda_l H + \mu_l\, F^2$ ,

provides a direct mechanism for incorporating curvature potentials into differential geometric models. Its non-vanishing (Theorem 2.8) is relevant for:

a) constructing anisotropic Lagrangians,
b) analysing geometric energy conditions,
c) and developing curvature-informed dynamical systems.

### 4.5. Computational and Modelling Implications

The structural identities (3.9)-(3.12) furnish implementable differential constraints that:

a) simplify symbolic computation in Finsler geometry,
b) support numerical simulation of anisotropic geodesics,
c) and provide stable geometric priors in computational models.

## 5. Conclusions

In this paper, we investigated the geometric structure of generalized $H^{v}$-recurrent Finsler spaces $GH^{v}\text{-}RF_n$, characterized by a Berwald curvature tensor whose $\boldsymbol{v}$-covariant derivative satisfies a two-vector recurrence condition. By employing Cartan's first-kind covariant differential operator and the fundamental identities of Finsler geometry, we established a comprehensive set of structural results governing the behavior of Berwald-type tensors under recurrence.

The study revealed that the recurrence condition imposed on the Berwald curvature tensor necessarily induces corresponding recurrence relations for several associated geometric quantities. In particular, we proved that the torsion tensor, the deviation tensor, the Berwald Ricci tensor, the curvature vector, and the Berwald scalar curvature all satisfy first-order $v$-covariant recurrence relations. As a significant consequence, it was shown that the Berwald Ricci tensor and the Berwald scalar curvature are non-vanishing in any generalized $H^v$-recurrent Finsler space, establishing an intrinsic rigidity within this geometric structure.

Furthermore, we demonstrated that the recurrence condition is equivalently characterized through alternative tensorial relations, notably condition (2.7), which provides a complete characterization of the $GH^v$-$RF_n$spaces. The first-order $v$-covariant derivative of the associated h(v)-torsion tensor was also shown to be uniquely determined by the corresponding recurrence condition.

A key contribution of this work is the derivation of necessary and sufficient conditions for the generalized recurrence of several fundamental tensors, including the Berwald curvature tensor, the Ricci-type tensor, and Izumi's tensor $Z^i_{jkh}$. These results were obtained through successive differentiation with respect to directional arguments and the application of commutation formulas associated with the Berwald connection. The resulting identities, such as conditions (3.3), (3.6), and the recurrence criterion for $Z^i_{jkh}$, provide a unified framework for understanding recurrence phenomena in Finsler spaces of this type.

Additionally, the investigation established structural identities for the associate curvature tensor of the Berwald curvature, including the symmetry relation (2.14), which further elucidates the internal geometry of generalized $H^v$-recurrent spaces.

Overall, the results obtained demonstrate that $GH^v$-$RF_n$spaces possess a rich geometric structure governed by interconnected recurrence relations across multiple curvature tensors. These findings contribute to the broader theory of Finsler spaces with special curvature properties and provide a foundation for future research on generalized recurrence, invariant structures, and potential applications in geometric analysis and theoretical physics.

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# الخصائص الهندسية وشروط التكرار في الفضاءات الفنزلرية المعممة -$H^v$ متكررة مع تطبيقات في النمذجة الهندسية التفاضلية

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**الملخص:** يتناول هذا البحث دراسة البنية الهندسية وخصائص التكرارية في الفضاءات الفنزلرية المعممة من النمط $GH^v$-$RF_n$، والمميزة بوجود موتر تقوس بيروالد $H^i_{jkh}$ الذي يحقق شرط تكرارية محدد تحت المشتق الترافقي من النوع $v$. يتم اشتقاق عدة علاقات أساسية للمشتقات الترافقية من الدرجة الأولى لموتر تقوس بيروالد، وموتر الانحراف، وموتر ريتشي، ومتجه التقوس، إضافة إلى موترات الليّ والانحناء المرتبطة بها. كما يتم الحصول على الشروط الضرورية والكافية لتكون هذه الموترات ذات تكرارية معممة، وذلك من خلال تحليل تفصيلي باستخدام معاملي كارتان الترافقِيَّين من النوعين الأول والثاني.

وتُظهر النتائج أن موتر ريتشي لبيروالد والدرجة القياسية لتقوس بيروالد لا يمكن أن يختفيا في هذا النوع من الفضاءات، وأن العديد من الكيانات الهندسية تُبدي سلوكًا تكراريًا صارمًا تحكمه متجهات التكرار $\lambda_l$ و$\mu_l$. علاوة على ذلك، تم الحصول على شروط التكرارية المعممة لموتر إيزومي $Z^i_{jkh}$، إلى جانب اشتقاق هويات جديدة تربط بين مكوّنات مختلفة من موترات الليّ والتقوس.

يوفّر الإطار النظري الذي تم تطويره في هذا البحث فهمًا أعمق لبنية الفضاءات الفنزلرية ذات التكرارية المعممة، ويدعم التطبيقات المحتملة في المجالات الهندسية التي تعتمد على النمذجة الهندسية التفاضلية، مثل الأنظمة الميكانيكية غير الخطية، نمذجة المواد اللاخطية وغير المتجانسة، وأنظمة التحكم المتقدمة.

**الكلمات المفتاحية:** هندسة فنزلر، الفضاءات المعممة من النمط -$H^v$ متكررة، تقوس بيروالد، موترات التكرار، النمذجة الهندسية التفاضلية، المشتقات الترافقية، موتر إيزومي.