JEF/Journal of Education Faculties
Volume 19, Issue (1), 2025

مجلة كليات التربية - جامعة عدن
المجلد 19 ، العدد (1)، 2025 م

Generalized Bi-Recurrent Structures in $G^{2nd}C_{|h}$-RF_n Spaces via the Weyl Conformal Curvature Tensor

Adel Mohammed Ali Al-Qashbari [1,2]
[1]Dept. of Math's., Faculty of Educ., Univ. of Aden
[2]Dept. of Med. Eng., Faculty of the Eng. and Computers, Univ. of Science & Technology, Aden
Email: Adel_ma71@yahoo.com

Fahmi Ahmed Mothana AL-ssallal [3]
[3]Dept. of Math's., Faculty of Educ., Univ. of Aden
Email: fahmiassallald55@gmail.com

DOI: https://doi.org/ 10.47372/jef.(2025)19.1.135

Abstract: In this paper, we investigate the properties of the Weyl conformal curvature tensor C^i_{jkh} in the context of $n = 4$ Riemannian and Finslerian spaces, with a particular focus on generalized recurrent and birecurrent structures. We derive several equivalent forms of the conformal curvature tensor under various covariant derivatives, revealing deep interrelations between curvature tensors, Ricci tensors, scalar curvature, and their derivatives. By transvecting the conformal curvature expressions with vectors such as y^i, y^k, and tensors such as g_{ij} we deduce necessary and sufficient conditions for the conformal curvature tensor, torsion tensor, Ricci tensor, and projective deviation tensor to represent generalized recurrent and birecurrent Finsler spaces. The results culminate in a sequence of theorems (Theorems 3.1 to 3.8), offering a comprehensive characterization of $G^{2nd}C_{|h}$-RF_n spaces and $G^{2nd}C_{|h}$-BRF_n spaces. These findings contribute to the geometric understanding of recurrence structures in differential geometry and extend the theoretical framework of Finsler geometry.

Keywords: The h-Covariant derivative of first and second orders, Generalized recurrent Finsler space, Weyl tensor W^i_{jkh} and conformal tensor C^i_{jkh} .

1. Introduction: The concept of conformal curvature tensors, particularly the Weyl tensor, plays a significant role in the study of differential geometry and theoretical physics. In Riemannian geometry, the Weyl tensor characterizes the conformal properties of the manifold, providing a measure of the deviation from conformal flatness. Extending such structures to Finsler geometry a generalization of Riemannian geometry introduces rich geometrical complexity and broader curvature behaviors.

This study focuses on analyzing the behavior of the Weyl conformal curvature tensor C^i_{jkh} within the framework of Finsler spaces. In particular, we explore its recurrence properties under the context of generalized recurrent Finsler (RF) and birecurrent Finsler (BRF) spaces, denoted as $G^{2nd}C_{|h}$-RF_n and $G^{2nd}C_{|h}$-BRF_n, respectively. The work involves detailed covariant differentiation of the Weyl tensor, incorporation of scalar, Ricci, and projective tensors, and identification of conditions under which these tensors preserve generalized recurrence structures.

By establishing several theorems and proving equivalence conditions for recurrence behaviors, we provide a unified framework for understanding the conformal geometric structures in Finsler spaces. The theoretical insights gained from this research not only deepen the algebraic understanding of Finslerian recurrence but also set the groundwork for potential applications in modern geometric theories.

The study of curvature tensors in Riemannian and Finsler geometries has attracted considerable attention due to its fundamental role in understanding the intrinsic structure of manifolds and their generalizations. Over the years, numerous researchers have contributed to the development of this area by introducing various types of curvature tensors and analyzing their recurrence properties and geometric implications. Notably, the works of Abu-Donia et al. (2020), Ahsan and Ali (2014, 2016) explored the role of www- and w∗w^*w∗-curvature tensors in the context of relativistic space-times, highlighting their significance in mathematical physics. Further, Al-Qashbari and collaborators have made extensive contributions to the field by introducing generalized recurrent, birecurrent, and trirecurrent Finsler structures involving higher-order covariant derivatives, such as those of Berwald and Cartan (e.g., Al-Qashbari et al., 2024; 2025). Their studies on projective, conharmonic, and Weyl curvature tensors within the Finslerian framework have deepened the understanding of higher-order geometrical structures and their decomposition properties.

In addition, foundational works by Misra et al. (2014), Pandey et al. (2011), and Goswami (2017) laid theoretical groundwork for higher-order recurrence and its applications in specialized Finsler spaces. Meanwhile, Rund's classic monograph (1981) continues to serve as a cornerstone reference in the differential geometry of Finsler spaces.

This body of literature provides the essential theoretical framework and motivation for the present study, which aims to further investigate the behavior of the conformal curvature tensor in the setting of generalized recurrent and birecurrent Finsler spaces using higher-order covariant derivatives and associated curvature relations.

The work of Al-Qashbari and his colleagues, including their studies on generalized recurrent Finsler spaces and various decomposition techniques, contributes to the ongoing development of Finsler geometry. Their research on the conformal curvature tensor and its properties in generalized Finsler spaces provides valuable insights into the intricate relationships between curvature, torsion, and the underlying geometric structures.

This paper builds on the foundation laid by previous studies, particularly focusing on the role of conformal curvature tensors in generalized recurrent Finsler spaces. By extending existing methods and exploring new techniques, we aim to deepen the understanding of the geometry of these spaces, offering new avenues for further research in the field.

In this paper, we investigate some identities between Weyl's tensor W^i_{jkh} and conformal tensor C^i_{jkh}. We first introduce the basic concepts of Weyl's curvature tensor and conformal tensor C^i_{jkh}. Then, we derive some identities between these two tensors.

2. Preliminaries:

In this section, some conditions and definitions well be provided for the purpose of this paper. Two vectors y_i and y^i meet the following conditions

$$a)\quad y_i = g_{ij}\, y^j \;,\quad b)\quad y_i\, y^i = F^2 \,,\quad c)\quad \delta^k_j y^j = y^k \quad \text{and} \quad d)\ \dot{\partial}_j y_k = g_{jk} \;. \tag{2.1}$$

The quantities g_{ij} and g^{ij} are related by

$$a)\ g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 \;, & if \quad i = k \;, \\ 0 \;, & if \quad i \neq k \;. \end{cases}$$

$$b)\ g^{jk}{}_{|h} = 0 \;,\ c)\ g_{ij|h} = 0 \;,\ d)\ g_{ir}\, \delta^i_j = g_{rj} \quad \text{and} \quad e)\ g^{jk}\, \delta^i_k = g^{ji} \;. \tag{2.2}$$

The vector y^i and metric function F are vanished identically for Cartan's covariant derivative.

$$a)\ F_{|h} = 0 \quad \text{and} \quad b)\ y^i{}_{|h} = 0 \;. \tag{2.3}$$

The h-covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j, successively, we get

$$X^i_{|k|j} = \partial_j\left(X^i_{|k}\right) - \left(X^i_{|r}\right)\Gamma^{*r}_{kj} + \left(X^r_{|k}\right)\Gamma^{*i}_{rj} - \left(\partial_j X^i_{|k}\right)\Gamma^{*i}_{js}\, y^s. \tag{2.4}$$

Tensor W^i_{jkh} , torsion tensor W^i_{jk} and deviation tensor W^i_j are defined by:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)}\dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)}\left(n\,H_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}\right.$$
$$\left. - \frac{\delta^i_h}{(n^2-1)}\left(n\,H_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}\right) \quad , \tag{2.5}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\left\{ \frac{\delta^i_{[j}}{(n^2-1)} \left(n\, H_{k]} - y^r H_{k]\,r} \right) \right\} \quad \text{and} \tag{2.6}$$

$$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{(n+1)} \left(\dot{\partial}_r H^r_j - \dot{\partial}_j H \right) y^i, \text{ respectively.} \tag{2.7}$$

Also, if we suppose that the tensor W^i_j satisfy the following identities

a) $W^i_k\, y^k = 0$, b) $W^i_i = 0$, c) $W^i_k\, y_i = 0$,

d) $g_{ir}\, W^i_j = W_{rj}$, e) $g^{jk} W_{jk} = W$ and f) $W_{jk}\, y^k = 0$. (2.8)

The conformal tensor C^i_{jkh}, torsion tensor C^i_{jk}, Ricci tensor C_{jk}, curvature vector C_k and scalar curvature C are satisfying:

a) $C^i_{jkh}\, y^j = C^i_{kh}$, b) $C^i_{kh}\, y^k = C^i_h$, c) $C^i_{jki} = C_{jk}$

d) $C^i_{ki} = C_k$, e) $C^i_i = C$ and f) $g_{ir}\, C^i_{jkh} = C_{rjkh}$. (2.9)

Cartan's 3rd curvature tensor R^i_{jkh}, Ricci tensor R_{jk}, the vector H_k and scalar curvature H are defined as

a) $R_{jk}\, y^j = H_k$, b) $R_{jk}\, y^k = R_j$, c) $R^i_i = R$ and d) $H_k\, y^k = (n-1)H$. (2.10)

Let us consider a Finsler space F_n which the Weyl's projective curvature tensor W^i_{jkh} satisfies a generalization generalized $W_{|h}$-recurrent space and denoted by $G^{2nd}\, W_{|h} - RF_n$. i.e. satisfies the following condition [5 - 6]

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m \left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right) + \frac{1}{4} \gamma_m \left(W^i_h g_{jk} - W^i_k g_{jh} \right). \tag{2.11}$$

where λ_m, μ_m and γ_m are non-zero covariant vectors of first order.

By taking the $h-$covariant derivative of (2.11), with respect to x^l, we obtain:

$$W^i_{jkh|m|l} = a_{ml} W^i_{jkh} + b_{ml} \left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right) + \frac{1}{4} c_{ml} \left(W^i_h g_{jk} - W^i_k g_{jh} \right) + \frac{1}{4} \gamma_m \left(W^i_h g_{jk} - W^i_k g_{jh} \right)_{|l} . \tag{2.12}$$

where $a_{ml} = \lambda_{m|l} + \lambda_m \lambda_l$, $b_{ml} = \mu_{m|l} + \lambda_m \mu_l$ and $c_{ml} = (\lambda_m \gamma_l + \gamma_{m|l})$ are non-zero covariant tensors field of second order and γ_m is non-zero covariant victor of first order, respectively.

Definition 2.1. In Finsler space, which the Wely's projective curvature tensor W^i_{jkh} satisfies the condition (2.12) is called the generalization generalized $W_{|h}$-birecurrent space and the tensor will be called a generalization generalized h-birecurrent space. These space and tensor denote them briefly by $G^{2nd}\, W_{|h} - BRF_n$ and $G^{2nd}\, h - BR$, respectively.

A Finsler space F_n which the curvature tensor W^i_{jkh} satisfies the condition (2.12) is referred to as the generalization generalized $W_{|h}-$birecurrent space and denoted by $G^{2nd}\, W_{|h} - BRF_n$.

In next section, we introduce a new class of Finsler spaces, namely, generalized $C_{|h}$-recurrent spaces and generalized $C_{|h}$-birecurrent spaces. These spaces generalize the concept of recurrence and birecurrence to a broader setting and exhibit interesting geometric properties. We investigate the curvature tensor of these spaces and establish several characterization theorems.

3. Relationship Between Weyl's Curvature Tensor and Conformal Curvature Tensor

Finsler geometry, as a generalization of Riemannian geometry, provides a powerful framework for modeling a wide range of physical phenomena. In Finsler spaces, the curvature properties of the space are characterized by various curvature tensors, among which Weyl and the conformal curvature tensor C^i_{jkh} play a significant role. While the geometric interpretations and physical implications of these tensors have been extensively studied, the relationship between them remains a subject of ongoing research. This paper aims to investigate the connection between Weyl's curvature tensor and the conformal curvature tensor C^i_{jkh} in Finsler spaces. By exploring their algebraic and geometric properties, we seek to establish new identities and inequalities that relate these two tensors. Our findings are expected to contribute to a deeper understanding of the curvature structure of Finsler spaces and provide insights into their applications in physics, such as in the study of gravitational theories and cosmology.

Some properties of W^i_{jkh} curvature tensor was proposed by Al-Qashbari, Abdallah and Al-ssallal [4].

For $(n=4)$ a Riemannian space, Weyl defined the conformal curvature tensor C^i_{jkh} often known as the Weyl conformal curvature tensor, as

$$W^i_{jkh} = C^i_{jkh} + \frac{1}{2}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right) + \frac{5}{6}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) + \frac{R}{6}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right). \tag{3.1}$$

By taking the $h-$covariant derivative of (3.1), with respect to x^m, we obtain:

$$W^i_{jkh|m} = C^i_{jkh|m} + \frac{1}{2}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)_{|m} + \frac{5}{6}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m}$$
$$+\frac{1}{6}\left(R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)\right)_{|m}. \tag{3.2}$$

Using (2.2c), in the equation (3.2) can be written as

$$W^i_{jkh|m} = C^i_{jkh|m} + \frac{1}{2}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)_{|m} + \frac{5}{6}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m}$$
$$+\frac{1}{6}R_{|m}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right). \tag{3.3}$$

By substituting equations (2.11) and (3.1) into (3.3), we obtain:

$$C^i_{jkh|m} + \frac{1}{2}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)_{|m} + \frac{5}{6}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m} + \frac{1}{6}R_{|m}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$= \lambda_m C^i_{jkh} + \frac{1}{2}\lambda_m\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right) + \frac{5}{6}\lambda_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) + \frac{1}{6}\lambda_m R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$+\mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right). \tag{3.4}$$

Or, we write as

$$C^i_{jkh|m} = \lambda_m C^i_{jkh} + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right) - \frac{1}{2}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)_{|m}$$
$$-\frac{5}{6}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m} - \frac{1}{6}R_{|m}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{2}\lambda_m\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)$$
$$+\frac{5}{6}\lambda_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) + \frac{1}{6}\lambda_m R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right). \tag{3.5}$$

This demonstrates that

$$C^i_{jkh|m} = \lambda_m C^i_{jkh} + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \frac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right). \tag{3.6}$$

If and only if

$$\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)_{|m} = \frac{1}{2}\lambda_m\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right),$$
$$\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m} = \lambda_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) \text{ and}$$
$$R_{|m}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) = \lambda_m R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right). \tag{3.7}$$

In conclusion the proof of theorem is completed, we can determine

Theorem 3.1. In the space $G^{2nd}C_{|h} - RF_n$, the conformal curvature tensor C^i_{jkh} represents a generalized recurrent Finsler space, provided that the condition (3.7) is satisfied.

By transvecting the condition to a higher-dimensional space as given in equation (3.5) with respect to y^j, and utilizing relations (2.9a), (2.1a), (2.3b) and (2.10a), we obtain the following result

$$C^i_{kh|m} = \lambda_m C^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) + \frac{1}{4}\gamma_m\left(W^i_k y_h - W^i_h y_k\right) - \frac{1}{2}\left(y_h R^i_k - \delta^i_h H_k\right)_{|m}$$
$$-\frac{5}{6}\left(\delta^i_k H_h - y_k R^i_h\right)_{|m} - \frac{1}{6}R_{|m}\left(\delta^i_h y_k - \delta^i_k y_h\right) + \frac{1}{2}\lambda_m\left(y_h R^i_k - \delta^i_h H_k\right)$$
$$+\frac{5}{6}\lambda_m\left(\delta^i_k H_h - y_k R^i_h\right) + \frac{1}{6}\lambda_m R\left(\delta^i_h y_k - \delta^i_k y_h\right). \tag{3.8}$$

This demonstrates that

$$C^i_{kh|m} = \lambda_m C^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) + \frac{1}{4}\gamma_m\left(W^i_k y_h - W^i_h y_k\right). \tag{3.9}$$

If and only if

$$\left(y_h R^i_k - \delta^i_h H_k\right)_{|m} = \frac{1}{2}\lambda_m\left(y_h R^i_k - \delta^i_h H_k\right),$$
$$\left(\delta^i_k H_h - y_k R^i_h\right)_{|m} = \lambda_m\left(\delta^i_k H_h - y_k R^i_h\right) \text{ and}$$
$$R_{|m}\left(\delta^i_h y_k - \delta^i_k y_h\right) = \lambda_m R\left(\delta^i_h y_k - \delta^i_k y_h\right). \tag{3.10}$$

Therefore, the proof of theorem is completed, we conclude

Theorem 3.2. In the space $G^{2nd}C_{|h} - RF_n$, the torsion tensor C^i_{kh} (Conformal curvature tensor C^i_{jkh}) represents a generalized recurrent Finsler space, provided that the condition (3.10) is satisfied.

By transvecting the condition to a higher-dimensional space as given in equation (3.8) with respect to y^k, and utilizing relations ($n = 4$), (2.9b), (2.3a), (2.3b), (2.1b), (2.8a), (2.1c) and (2.10e), we obtain the following result

$$C^i_{h|m} = \lambda_m C^i_h + \mu_m(\delta^i_h F^2 - y^i y_h) + \frac{1}{4}\gamma_m[W^i_h F^2] - \frac{1}{2}(y_h R^i_k y^k - 3\delta^i_h H)_{|m}$$
$$-\frac{5}{6}(y^i H_h - F^2 R^i_h)_{|m} - \frac{1}{6}R_{|m}(\delta^i_h F^2 - y^i y_h) + \frac{1}{2}\lambda_m(y_h R^i_k y^k - 3\delta^i_h H)$$
$$+\frac{5}{6}\lambda_m(y^i H_h - F^2 R^i_h) + \frac{1}{6}\lambda_m R(\delta^i_h F^2 - y^i y_h) \quad . \tag{3.11}$$

This demonstrates that

$$C^i_{h|m} = \lambda_m C^i_h + \mu_m(\delta^i_h F^2 - y^i y_h) + \frac{1}{4}\gamma_m[W^i_h F^2]. \tag{3.12}$$

If and only if

$$(y_h R^i_k y^k - 3\delta^i_h H)_{|m} = \lambda_m(y_h R^i_k y^k - 3\delta^i_h H) \ ,$$
$$(y^i H_h - F^2 R^i_h)_{|m} = \lambda_m(y^i H_h - F^2 R^i_h) \ \text{and}$$
$$R_{|m}(\delta^i_h F^2 - y^i y_h) = \lambda_m R(\delta^i_h F^2 - y^i y_h) \, . \tag{3.13}$$

Therefore, the proof of theorem is completed, we conclude

Theorem 3.3. In the space $G^{2nd}C_{|h} - RF_n$, the projective deviation tensor C^i_h represents a generalized recurrent Finsler space if and only if the tensors $(y_h R^i_k y^k - 3\delta^i_h H), (y^i H_h - F^2 R^i_h)$ and $R(\delta^i_h F^2 - y^i y_h)$ are generalized recurrent Finsler space.

By contracting the index space through summation over indices i and h in the equations (3.5), (3.8) and (3.11), and applying relations (2.2a), (2.1a), (2.1b), (2.8b), (2.8c), (2.8d), (2.10c), (2.10d) and (2.1c), in view of (2.9c), (2.9d) and (2.9e), we obtain the following result

$$C_{jk|m} = \lambda_m C_{jk} + \mu_m(n-1)g_{jk} - \frac{1}{4}\gamma_m[W_{jk}] - \frac{1}{2}(1-n)R_{jk|m} - \frac{5}{6}(R_{jk} - g_{jk}R)_{|m}$$
$$-\frac{1}{6}R_{|m}(n-1)g_{jk} + \frac{1}{6}\lambda_m R(n-1)g_{jk} + \frac{1}{2}\lambda_m(1-n)R_{jk} + \frac{5}{6}\lambda_m(R_{jk} - g_{jk}R). \tag{3.14}$$

This demonstrates that

$$C_{jk|m} = \lambda_m C_{jk} + \mu_m(n-1)g_{jk} - \frac{1}{4}\gamma_m[W_{jk}]. \tag{3.15}$$

If and only if

$$R_{jk|m} = \lambda_m R_{jk},$$
$$(R_{jk} - g_{jk}R)_{|m} = \lambda_m(R_{jk} - g_{jk}R)$$
$$R_{|m} = \lambda_m R. \tag{3.16}$$

and

$$C_{k|m} = \lambda_m C_k + \mu_m(n-1)\, y_k - \frac{1}{2}(y_i R^i_k - nH_k)_{|m} - \frac{5}{6}(H_k - y_k R)_{|m} - \frac{1}{6}(n-1)\,(R y_k)_{|m}$$
$$+\frac{1}{2}\lambda_m(y_i R^i_k - nH_k) + \frac{5}{6}\lambda_m(H_k - y_k R) + \frac{1}{6}(n-1)\lambda_m R y_k \, . \tag{3.17}$$

This demonstrates that

$$C_{k|m} = \lambda_m C_k + \mu_m(n-1)\, y_k \, . \tag{3.18}$$

If and only if

$$(y_i R^i_k - nH_k)_{|m} = \lambda_m(y_i R^i_k - nH_k)$$
$$(H_k - y_k R)_{|m} = \lambda_m(H_k - y_k R)$$
$$(R y_k)_{|m} = \lambda_m R y_k \ . \tag{3.19}$$

In the last

$$C_{|m} = \lambda_m C + \mu_m (n-1) F^2 - \tfrac{1}{2}\left(y_i R^i_k y^k - 3nH\right)_{|m} - \tfrac{5}{6}(3H - F^2 R)_{|m} - \tfrac{(n-1)}{6}(RF^2)_{|m}$$
$$+\tfrac{1}{2}\lambda_m\left(y_h R^i_k y^k - 3nH\right) + \tfrac{5}{6}\lambda_m(3H - F^2 R) + \tfrac{1}{6}(n-1)\lambda_m R F^2. \quad (3.20)$$

This demonstrates that

$$C_{|m} = \lambda_m C + \mu_m (n-1) F^2. \quad (3.21)$$

If and only if

$$\left(y_i R^i_k y^k - 3nH\right)_{|m} = \lambda_m\left(y_h R^i_k y^k - 3nH\right),$$
$$(3H - F^2 R)_{|m} = \lambda_m (3H - F^2 R) \quad \text{and}$$
$$(RF^2)_{|m} = \lambda_m R F^2. \quad (3.22)$$

In conclusion the proof of theorem is completed, we can say

Theorem 3.4. In the space $G^{2nd}C_{|h} - RF_n$, Ricci tensor C_{jk} , vector C_k and scalar C are defined in equations (3.15), (3.18) and (3.21) , respectively, if and only if the conditions (3.16), (3.19) and (3.22) are satisfied.

By transvecting the condition to a higher-dimensional space as given in equation (3.5) with respect to g_{ir}, and utilizing relations (2.1d), (2.2c), (2.8d) and (2.9f), we obtain the following result

$$C_{rjkh|m} = \lambda_m C_{rjkh} + \mu_m\left(g_{rh}g_{jk} - g_{rk}g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W_{rh}g_{jk} - W_{rk}g_{jh}\right] - \tfrac{1}{2}\left(g_{jh}R_{rk} - g_{rh}R_{jk}\right)_{|m}$$
$$-\tfrac{5}{6}\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right)_{|m} - \tfrac{1}{6}\left(R(g_{rh}g_{jk} - g_{rk}g_{jh})\right)_{|m} + \tfrac{1}{2}\lambda_m\left(g_{jh}R_{rk} - g_{rh}R_{jk}\right)$$
$$+\tfrac{5}{6}\lambda_m\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right) + \tfrac{1}{6}\lambda_m R\left(g_{rh}g_{jk} - g_{rk}g_{jh}\right). \quad (3.23)$$

This demonstrates that

$$C_{rjkh|m} = \lambda_m C_{rjkh} + \mu_m\left(g_{rh}g_{jk} - g_{rk}g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W_{rh}g_{jk} - W_{rk}g_{jh}\right]. \quad (3.24)$$

If and only if

$$\left(g_{jh}R_{rk} - g_{rh}R_{jk}\right)_{|m} = \lambda_m\left(g_{jh}R_{rk} - g_{rh}R_{jk}\right),$$
$$\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right)_{|m} = \lambda_m\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right) \text{ and}$$
$$\left(R(g_{rh}g_{jk} - g_{rk}g_{jh})\right)_{|m} = \lambda_m R\left(g_{rh}g_{jk} - g_{rk}g_{jh}\right). \quad (3.25)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.5. In the space $G^{2nd}C_{|h} - RF_n$, associate tensor C_{jrkh} (Conformal curvature tensor C^i_{jkh}) represents a generalized recurrent Finsler space, provided that the condition (3.25) is satisfied.

By taking the $h-$covariant derivative of (3.1), with respect to x^m and x^l , respectively, and applying the condition (2.2c), we get

$$W^i_{jkh|m|l} = C^i_{jkh|m|l} + \tfrac{1}{2}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)_{|m|l} + \tfrac{5}{6}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)_{|m|l}$$
$$+\tfrac{1}{6}\left(R(\delta^i_h g_{jk} - \delta^i_k g_{jh})\right)_{|m|l}. \quad (3.26)$$

By substituting equations (2.12) and (3.1) into (3.26), we obtain:

$$C^i_{jkh|m|l} + \tfrac{1}{2}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)_{|m|l} + \tfrac{5}{6}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)_{|m|l} + \tfrac{1}{6}\left(R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)\right)_{|m|l}$$
$$= a_{ml}C^i_{jkh} + \tfrac{1}{2}a_{ml}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right) + \tfrac{5}{6}a_{ml}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right) + \tfrac{1}{6}a_{ml}R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$+b_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l}.$$

Or, we write as

$$C^i_{jkh|m|l} = a_{ml}C^i_{jkh} + b_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)$$
$$+\tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} - \tfrac{1}{2}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right)_{|m|l} - \tfrac{5}{6}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)_{|m|l}$$
$$-\tfrac{1}{6}\left(R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)\right)_{|m|l} + \tfrac{1}{2}a_{ml}\left(g_{jh}R^i_k - \delta^i_h R_{jk}\right) + \tfrac{5}{6}a_{ml}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)$$

$$+\frac{1}{6}a_{ml}\mathrm{R}(\delta^i_h\, g_{jk}-\delta^i_k\, g_{jh}). \quad (3.27)$$

This demonstrates that

$$\mathrm{C}^i_{jkh|m|l}=a_{ml}\mathrm{C}^i_{jkh}+b_{ml}(\delta^i_h g_{jk}-\delta^i_k g_{jh})+\frac{1}{4}c_{ml}(W^i_h g_{jk}-W^i_k g_{jh})$$
$$+\frac{1}{4}\gamma_m(W^i_h g_{jk}-W^i_k g_{jh})_{|l}\ . \quad (3.28)$$

If and only if

$$(g_{jh}R^i_k-\delta^i_h\, R_{jk})_{|m|l}=a_{ml}(g_{jh}R^i_k-\delta^i_h\, R_{jk}),$$
$$(\delta^i_k\, R_{jh}-\ g_{jk}\, R^i_h)_{|m|l}=a_{ml}(\delta^i_k\, R_{jh}-\ g_{jk}\, R^i_h)\ \text{ and}$$
$$\left(\mathrm{R}(\delta^i_h\, g_{jk}-\delta^i_k\, g_{jh})\right)_{|m|l}=a_{ml}\mathrm{R}(\delta^i_h\, g_{jk}-\delta^i_k\, g_{jh}). \quad (3.29)$$

In conclusion the proof of theorem is completed, we can determine

Theorem 3.6. In the space $G^{2nd}C_{|h}-BRF_n$, conformal curvature tensor C^i_{jkh} represents a generalized birecurrent Finsler space, provided that the condition (3.29) is satisfied.

By transvecting the condition to a higher-dimensional space as given in equation (3.27) with respect to y^j, and utilizing relations (2.9a), (2.1a), (2.3b) and (2.10a), we obtain the following result

$$\mathrm{C}^i_{kh|m|l}=a_{ml}\mathrm{C}^i_{kh}+b_{ml}(\delta^i_h y_k-\delta^i_k y_h)+\frac{1}{4}c_{ml}(W^i_h y_k-W^i_k y_h)+\frac{1}{4}\gamma_m(W^i_h y_k-W^i_k y_h)_{|l}$$
$$-\frac{1}{2}(y_h R^i_k-\delta^i_h H_k)_{|m|l}-\frac{5}{6}(\delta^i_k H_h-y_k R^i_h)_{|m|l}-\frac{1}{6}\left(\mathrm{R}(\delta^i_h y_k-\delta^i_k y_h)\right)_{|m|l}+\frac{1}{2}a_{ml}(y_h R^i_k-\delta^i_h H_k)$$
$$+\frac{5}{6}a_{ml}(\delta^i_k\, H_h-\ y_k R^i_h)+\frac{1}{6}a_{ml}\mathrm{R}(\delta^i_h y_k-\delta^i_k y_h)\ . \quad (3.30)$$

This demonstrates that

$$\mathrm{C}^i_{kh|m|l}=a_{ml}\mathrm{C}^i_{kh}+b_{ml}(\delta^i_h y_k-\delta^i_k y_h)+\frac{1}{4}c_{ml}(W^i_h y_k-W^i_k y_h)$$
$$+\frac{1}{4}\gamma_m(W^i_h y_k-W^i_k y_h)_{|l}.$$

(3.31)

If and only if

$$(y_h R^i_k-\delta^i_h\, H_k)_{|m|l}=a_{ml}(y_h R^i_k-\delta^i_h\, H_k),$$
$$(\delta^i_k\, H_h-\ y_k R^i_h)_{|m|l}=a_{ml}(\delta^i_k\, H_h-\ y_k R^i_h)\ \text{ and}$$
$$\left(\mathrm{R}(\delta^i_h y_k-\delta^i_k y_h)\right)_{|m|l}=a_{ml}\mathrm{R}(\delta^i_h y_k-\delta^i_k y_h)\,. \quad (3.32)$$

Therefore, the proof of theorem is completed, we conclude

Theorem 3.7. In the space $G^{2nd}C_{|h}-BRF_n$, the torsion tensor C^i_{kh} (Conformal curvature tensor C^i_{jkh}) represents a generalized birecurrent Finsler space, provided that the condition (3.32) is satisfied.

By transvecting the condition to a higher-dimensional space as given in equation (3.30) with respect to y^k, and utilizing relations ($n=4$), (2.9b), (2.3a), (2.3b), (2.1b), (2.8a), (2.1c) and (2.10d), we obtain the following result

$$\mathrm{C}^i_{h|m|l}=a_{ml}\mathrm{C}^i_h+b_{ml}(\delta^i_h F^2-y^i y_h)+\frac{1}{4}c_{ml}(W^i_h F^2)+\frac{1}{4}\gamma_m(W^i_h F^2)_{|l}$$
$$-\frac{1}{2}(y_h R^i_k\, y^k-3\delta^i_h\, H)_{|m|l}-\frac{5}{6}(y^i H_h-F^2 R^i_h)_{|m|l}-\frac{1}{6}\left(\mathrm{R}(\delta^i_h F^2-y^i y_h)\right)_{|m|l}$$
$$+\frac{1}{2}a_{ml}(y_h R^i_k\, y^k-3\delta^i_h\, H)+\frac{5}{6}a_{ml}(y^i H_h-F^2 R^i_h)+\frac{1}{6}a_{ml}\mathrm{R}(\delta^i_h F^2-y^i y_h)\,. \quad (3.33)$$

This demonstrates that

$$\mathrm{C}^i_{h|m|l}=a_{ml}\mathrm{C}^i_h+b_{ml}(\delta^i_h F^2-y^i y_h)+\frac{1}{4}c_{ml}(W^i_h F^2)+\frac{1}{4}\gamma_m(W^i_h F^2)_{|l}. \quad (3.34)$$

If and only if

$$(y_h R^i_k\, y^k-3\delta^i_h\, H)_{|m|l}=a_{ml}(y_h R^i_k\, y^k-3\delta^i_h\, H)\,,$$
$$(y^i H_h-F^2 R^i_h)_{|m|l}=a_{ml}(y^i H_h-F^2 R^i_h)\ \text{ and}$$

$$\left(\mathrm{R}\left(\delta_h^i F^2 - y^i y_h\right)\right)_{|m|l} = a_{ml}\mathrm{R}\left(\delta_h^i F^2 - y^i y_h\right). \tag{3.35}$$

Therefore, the proof of theorem is completed, we conclude

Theorem 3.8. In the space $G^{2nd}C_{|h} - BRF_n$, the projective deviation tensor C_h^i represents a generalized birecurrent Finsler space if and only if the tensors $\left(y_h R_k^i\, y^k - 3\delta_h^i\, H\right)$, $\left(\, y^i H_h - F^2 R_h^i\,\right)$ and $R\left(\delta_h^i F^2 - y^i y_h\right)$ are generalized birecurrent Finsler space.

By contracting the index space through summation over indices i and h in the condition (3.27), (3.30) and (3.33), and applying relations (n=4), (2.2a), (2.1a), (2.1b), (2.8b), (2.8c), (2.8d), (2.10c), (2.10d) and (2.1c), in view of (2.9c), (2.9d) and (2.9e), we obtain the following result

$$\begin{aligned}\mathrm{C}_{jk|m|l} &= a_{ml}\mathrm{C}_{jk} + b_{ml}(n-1)g_{jk} + \tfrac{1}{4}c_{ml}W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l} - \tfrac{1}{2}\left((1-n)R_{jk}\right)_{|m|l} \\ &- \tfrac{5}{6}\left(\, R_{jk} - \, g_{jk}\,\mathrm{R}\right)_{|m|l} - \tfrac{1}{6}\left(\mathrm{R}(n-1)g_{jk}\right)_{|m|l} + \tfrac{1}{2}a_{ml}(1-n)R_{jk} + \tfrac{5}{6}a_{ml}\left(\, R_{jk} - \, g_{jk}\,\mathrm{R}\right) \\ &+ \tfrac{1}{6}a_{ml}\mathrm{R}(n-1)g_{jk}.\end{aligned} \tag{3.36}$$

This demonstrates that

$$\mathrm{C}_{jk|m|l} = a_{ml}\mathrm{C}_{jk} + b_{ml}(n-1)g_{jk} + \tfrac{1}{4}c_{ml}W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l}. \tag{3.37}$$

If and only if

$$\begin{aligned}&\left((1-n)R_{jk}\right)_{|m|l} = a_{ml}\left((1-n)R_{jk}\right), \\ &\left(\, R_{jk} - \, g_{jk}\,\mathrm{R}\right)_{|m|l} = a_{ml}\left(\, R_{jk} - \, g_{jk}\,\mathrm{R}\right) \text{ and} \\ &\left(\mathrm{R}(n-1)g_{jk}\right)_{|m|l} = a_{ml}\mathrm{R}(n-1)g_{jk}.\end{aligned} \tag{3.38}$$

And

$$\begin{aligned}\mathrm{C}_{k|m|l} &= a_{ml}\mathrm{C}_k + b_{ml}(n-1)\, y_k - \tfrac{1}{2}\left(y_i R_k^i - nH_k\right)_{|m|l} - \tfrac{5}{6}\left(H_k - \, y_k\mathrm{R}\right)_{|m|l} \\ &- \tfrac{1}{6}\left(\mathrm{R}(n-1)\, y_k\right)_{|m|l} + \tfrac{1}{2}a_{ml}\left(y_i R_k^i - nH_k\right) + \tfrac{5}{6}a_{ml}\left(H_k - \, y_k\mathrm{R}\right) + \tfrac{1}{6}a_{ml}\mathrm{R}(n-1)\, y_k.\end{aligned} \tag{3.39}$$

This demonstrates that

$$\mathrm{C}_{k|m|l} = a_{ml}\mathrm{C}_k + b_{ml}(n-1)\, y_k. \tag{3.40}$$

If and only if

$$\begin{aligned}&\left(y_i R_k^i - nH_k\right)_{|m|l} = a_{ml}\left(y_i R_k^i - nH_k\right), \\ &\left(H_k - \, y_k\mathrm{R}\right)_{|m|l} = a_{ml}\left(H_k - \, y_k\mathrm{R}\right) \text{ and} \\ &\left(R(n-1)\, y_k\right)_{|m|l} = a_{ml}R(n-1)\, y_k.\end{aligned} \tag{3.41}$$

In the last

$$\begin{aligned}\mathrm{C}_{|m|l} &= a_{ml}\mathrm{C} + b_{ml}(n-1)F^2 - \tfrac{1}{2}\left(y_i R_k^i\, y^k - 3nH\right)_{|m|l} - \tfrac{5}{6}\left(3H - \, F^2\mathrm{R}\right)_{|m|l} \\ &- \tfrac{1}{6}(n-1)\left(\mathrm{R}F^2\right)_{|m|l} + \tfrac{1}{2}a_{ml}\left(y_i R_k^i\, y^k - 3nH\right) + \tfrac{5}{6}a_{ml}\left(3H - F^2\mathrm{R}\right) \\ &+ \tfrac{1}{6}a_{ml}(n-1)\mathrm{R}F^2.\end{aligned} \tag{3.42}$$

This demonstrates that

$$\mathrm{C}_{|m|l} = a_{ml}\mathrm{C} + b_{ml}(n-1)F^2. \tag{3.43}$$

If and only if

$$\begin{aligned}&\left(y_i R_k^i\, y^k - 3nH\right)_{|m|l} = a_{ml}\left(y_i R_k^i\, y^k - 3nH\right), \\ &\left(3H - \, F^2\mathrm{R}\right)_{|m|l} = a_{ml}\left(3H - \, F^2\mathrm{R}\right) \text{ and} \\ &\left(RF^2\right)_{|m|l} = a_{ml}RF^2.\end{aligned} \tag{3.44}$$

In conclusion the proof of theorem is completed, we can say

Theorem 3.9. In the space $G^{2nd}C_{|h} - BRF_n$, Ricci tensor C_{jk} , vector C_k and scalar C are defined in equations (3.37), (3.40) and (3.43), respectively, provided that the conditions (3.39), (3.41) and (3.44) are satisfied.

By transvecting the condition to a higher-dimensional space as given in equation (3.27) with respect to g_{ir}, and utilizing relations (2.2c), (2.2d), (2.8d), and (2.9f), we obtain the following result

$$C_{rjkh|m|l} = a_{ml}C_{rjkh} + b_{ml}(g_{rh}g_{jk} - g_{rk}g_{jh}) + \frac{1}{4}c_{ml}(W_{rh}g_{jk} - W_{rk}g_{jh})$$
$$+\frac{1}{4}\gamma_m(W_{rh}g_{jk} - W_{rk}g_{jh})_{|l} - \frac{1}{2}(g_{jh}R_{rk} - g_{rh}R_{jk})_{|m|l} - \frac{5}{6}(g_{rk}R_{jh} - g_{jk}R_{rh})_{|m|l}$$
$$-\frac{1}{6}\left(R(g_{rh}g_{jk} - g_{rk}g_{jh})\right)_{|m|l} + \frac{1}{2}a_{ml}(g_{jh}R_{rk} - g_{rh}R_{jk}) + \frac{5}{6}a_{ml}(g_{rk}R_{jh} - g_{jk}R_{rh})$$
$$+\frac{1}{6}a_{ml}R(g_{rh}g_{jk} - g_{rk}g_{jh}) \ . \quad (3.45)$$

This demonstrates that

$$C_{rjkh|m|l} = a_{ml}C_{rjkh} + b_{ml}(g_{rh}g_{jk} - g_{rk}g_{jh}) + \frac{1}{4}c_{ml}(W_{rh}g_{jk} - W_{rk}g_{jh})$$
$$+\frac{1}{4}\gamma_m(W_{rh}g_{jk} - W_{rk}g_{jh})_{|l}. \quad (3.46)$$

If and only if

$$(g_{jh}R_{rk} - g_{rh}R_{jk})_{|m|l} = a_{ml}(g_{jh}R_{rk} - g_{rh}R_{jk}),$$
$$(g_{rk}R_{jh} - g_{jk}R_{rh})_{|m|l} = a_{ml}(g_{rk}R_{jh} - g_{jk}R_{rh}) \text{ and}$$
$$\left(R(g_{rh}g_{jk} - g_{rk}g_{jh})\right)_{|m|l} = a_{ml}R(g_{rh}g_{jk} - g_{rk}g_{jh}) \ . \quad (3.47)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.10. In the space $G^{2nd}C_{|h} - BRF_n$, associate tensor C_{jrkh} (Conformal curvature tensor C^i_{jkh}) represents a generalized birecurrent Finsler space, provided that the condition (3.47) is satisfied.

4. Conclusions

In this work, we have established a comprehensive treatment of the Weyl conformal curvature tensor C^i_{jkh} and its derivatives in the context of generalized recurrent and birecurrent Finsler spaces. Through rigorous covariant differentiation and tensor contractions, we derived multiple characterizations of the conformal, torsion, Ricci, and projective deviation tensors.

Our analysis led to the formulation of eight theorems (Theorems 3.1 to 3.8), each providing necessary and sufficient conditions for a specific tensor to represent a generalized recurrent or birecurrent structure in the Finsler space $G^{2nd}C_{|h}$-RF_n and $G^{2nd}C_{|h}$-BRF_n. These results highlight the critical role of conformal geometry in characterizing the intrinsic structure of Finsler spaces and contribute significantly to the broader field of differential geometry.

Future studies may build upon this framework to investigate applications in gravitational theories, spacetime models in physics, and further generalizations in Finsler and pseudo-Finsler geometry. The findings also suggest potential for exploring higher-order recurrence conditions and their geometric implications.

5. Recommendations

Based on the results of this research, we recommend the following directions for future research:

- Explore other types of decompositions: Investigate different decomposition schemes and their corresponding geometric interpretations.
- Investigate the physical implications: Explore the physical implications of the decomposition results, particularly in the context of field theories and cosmology.
- Develop numerical methods: Develop numerical methods for computing the decomposed tensors and analyzing their properties.

References

1. Abu-Donia, H., Shenawy, S. & Abdehameed, A. (2020). The w*-curvature tensor on relativistic space-times. Kyungpook Mathematical Journal, 60, 185-195.
2. Ahsan, Z. & Ali, M. (2014). On some properties of w-curvature tensor. Palestine Journal of Mathematics, 3 (1), 61-69.

3. Ahsan, Z. & Ali, M. (2016). Curvature tensor for the spacetime of general relativity. Palestine Journal of Mathematics, 2,1-15.
4. Al-Qashbari, A.M. Abdallah, A.A. and Al-ssallal, F.A. (2024). Recurrent Finsler structures with higher-order generalizations defined by special curvature tensors, International Journal of Advanced Research in Science, Communication and Technology,Vol.4, No 1,68-75,
5. Al-Qashbari, A.M. and Al-ssallal, F.A. (2024). Study of curvature tensor by using Berwald's and Cartan's higher-order derivatives in Finsler space, Technological Applied and Humanitarian Academic Journal, Vol.1, No 1, 1-15,
6. Al-Qashbari, A.M. Haoues M. and Al-ssallal, F.A. (2024). A decomposition analysis of Weyl's curvature tensor via Berwald's first and second order derivatives in Finsler spaces, Journal of Innovativ Applied Mathematics and Computational Science,4(2),201-213.
7. Al-Qashbari, A.M. Abdallah A.A. and Baleedi, S.M. (2025): Berwald covariant derivative and Lie derivative of conharmonic curvature tensors in generalized fifth recurrent Finsler space, GPH-International Journal of Mathematics, Vol.8, Issue 1, 24-32.
8. AL-Qashbari, A.M.A. & Qasem, F.Y.A. (2017). Study on generalized $\mathcal{B}$R-trirecurrent Finsler Space. Journal of Yemen Engineer,15, 79-89.
9. AL-Qashbari, A.M.A. (2020). On generalized for curvature tensors P^i_{jkh} of second order in Finsler space. University of Aden Journal of Natural and Applied Sciences, 24 (1), 171-176.
10. AL-Qashbari, A.M.A. (2019). Some properties for Weyl's projective curvature tensors of generalized W^h-birecurrent in Finsler spaces. University of Aden Journal of Natural and Applied Sciences,23, (1), 181-189.
11. AL-Qashbari, A.M.A. (2020). Some identities for generalized curvature tensors in $\mathcal{B}$-Recurrent Finsler space. Journal of New Theory, ISSN:2149-1402, 32. 30-39.
12. AL-Qashbari, A.M.A. (2020). Recurrence decompositions in Finsler space. Journal of Mathematical Analysis and Modeling, ISSN:2709-5924, 1, 77-86.
13. AL-Qashbari, A.M.A. & AL-Maisary A.A.S. (2023). Study on generalized W^i_{jkh} of fourth order recurrent in Finsler space. Journal of Yemeni Engineer, Univ. Aden, 17 (2), 72-86.
14. AL-Qashbari, A.M.A., Abdallah, A.A. & Nasr, K.S. (2025): On generalized Trirecurrent space by using Gh-Covariant derivative in Finsler geometry, Journal of Mathematical Problems, Equations and Statistics,Vol.6, No.1, 91-100.
15. AL-Qashbari, A.M.A., Saleh, S. and Ibedou, I. (2024): On some relations of R-projective curvature tensor in recurrent Finsler space, Journal of Non-Linear Modeling and Analysis (JNMA), (China), Vol. 6, No.4, 1216-1227.
16. A. M. A. Al-Qashbari, A. A. Abdallah, and S.M. Baleedi,"A Study of M-Projective Curvature Tensor $\bar{W}^i_{jkh}$ in $G\mathcal{B}K-5RF_n$ Via Lie Derivative", International Journal of Applied Science and Mathematical Theory, Vol.11, No1,1-9, 2025.
17. A. M. A. Al-Qashbari, A. A. Abdallah, and F. Abdallah, "Decomposition of Generalized Recurrent Tensor Fields of R^h-nth Order in Finsler Manifolds", Journal of Science and Technology, Vol.30, Issue 2, 99-105, 2025.
18. A. M. A. Al-Qashbari, A. A. Abdallah, and S.M. Baleedi,"Berwald covariant derivative and Lie derivative of conharmonic curvature tensors in generalized fifth recurrent Finsler space", GPH-International Journal of Mathematics, Vol.8, Issue 1, 24-32, 2025.
19. A. M. A. Al-Qashbari, A. A. Abdallah, and K.S. Nasr,"Generalized of h-torsion and curvature structures in generalized recurrent space of third order by Cartan covariant derivatives", GPH-International Journal of Mathematics, Vol.8, Issue 2, 1-9, 2025.
20. Baleedi, S.M.S. (2017). On certain generalized $\mathcal{B}$K-recurrent Finsler space (M. Sc. Thesis, University of Aden, Aden, Yemen).

21. Misra, B., Misra, S. B., Srivastava, K. & Srivastava, R. B. (2014). Higher order recurrent Finsler spaces with Berwald's curvature tensor field. Journal of Chemical, Biological and Physical Sciences, 4, (1), 624-631.
22. Goswami, A. (2017). A study of certain types of special Finsler spaces in differential geometry: Systematic Review. Journal of Mathematics Applied in Science and Technology, 9, (1), 23-30. Issn. 0973-6344,
23. Pandey, P.N., Saxena, S. & Goswani, A. (2011). On a generalized H-recurrent space. Journal of International Academy of Physical Sciences, 15, 201-211.
24. Rund, H. (1981). The differential geometry of Finsler spaces. Springer-Verlag, Berlin Göttingen-Heidelberg, 2nd Edit. (in Russian), Nauka, (Moscow).

الهياكل ثنائية التكرار المعممة في الفضاءات $G^{2nd}C_{|h}$-RF_n باستخدام موتر الانحناء التشاكلي لويل

عادل محمد علي القشبري [2،1]
قسم الرياضيات، كلية التربية عدن، جامعة عدن
قسم الهندسة الطبية- كلية الهندسة والحاسبات- جامعة العلوم والتكنولوجيا - عدن

فهمي أحمد مثنى السلال[3]
قسم الرياضيات، كلية التربية الضالع، جامعة عدن

الملخص: في هذا البحث، ندرس خصائص موتر الانحناء التشاكلي لويل C^i_{jkh} ضمن سياق الفضاءات الريمانية والفينسلرية لأبعاد $n = 4$ ، مع تركيز خاص على الهياكل المتكررة والمعممة ثنائية التكرار. قمنا باشتقاق عدة صيغ مكافئة لموتر الانحناء التشاكلي تحت اشتقاقات تغايرية مختلفة، مما كشف عن علاقات متبادلة عميقة بين موترات الانحناء، وموتر ريتشي، والانحناء القياسي ومشتقاتها. من خلال تحويل تعبيرات الانحناء التشاكلي باستخدام متجهات مثل y^i و y^k وموترات مثل g_{ij} ، استنتجنا الشروط الضرورية والكافية التي تمكّن موتر الانحناء التشاكلي، وموتر الالتواء، وموتر ريتشي، وموتر الانحراف الإسقاطي من تمثيل الفضاءات الفينسلرية المعممة والمتكررة ثنائياً. وقد توّجت النتائج بسلسلة من النظريات (من النظرية 3.1 إلى النظرية 3.8)، والتي تقدم توصيفاً شاملاً للفضاءات $G^{2nd}C_{|h}$-RF_n و $G^{2nd}C_{|h}$-BRF_n تسهم هذه النتائج في تعميق الفهم الهندسي لهياكل التكرار في الهندسة التفاضلية، كما توسّع الإطار النظري للهندسة الفينسلرية.

الكلمات المفتاحية: الاشتقاق التغايري من الرتبتين الأولى والثانية، فضاء فينسلري متكرر معمماً، موتر ويل W^i_{jkh}، وموتر الانحناء التشاكلي C^i_{jkh} .