Structure Properties and Fundamental Identities of Generalized R-Recurrent Finsler Manifolds

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DOI: https://doi.org/10.47372/jef.(2025)19.1.116

Abstract: In this paper, we introduce and investigate a new class of Finsler spaces, termed generalized R^h-recurrent Finsler spaces, denoted by GR^h-RF_n. These are defined via a generalized recurrence condition imposed on Cartan's third curvature tensor, involving three non-null covariant vector fields. We derive the fundamental characterizations of such spaces and establish their equivalence through multiple tensorial identities. The behavior of related geometric objects such as the h(v)-torsion tensor, Ricci tensor, curvature vector, deviation tensor, and scalar curvature is analyzed. Furthermore, several non-trivial identities involving covariant derivatives and contractions are proven, demonstrating rich internal symmetries. The study concludes with a series of structural theorems that extend classical recurrence concepts in Finsler geometry.

Keywords: Finsler geometry, Generalized recurrent spaces, R^h-recurrence , Curvature tensor, Covariant derivatives**.**

1. Introduction: Finsler geometry, as a natural generalization of Riemannian geometry, allows for the investigation of more intricate curvature structures and tensorial behaviors. Within this broader framework, recurrence conditions on curvature tensors have long served as a cornerstone for understanding the intrinsic properties of the space. Classical notions such as recurrent and R-recurrent spaces have been widely studied; however, their generalizations in the context of Finsler spaces offer a deeper exploration into the interplay of curvature, torsion, and directional dependence.

In this study, we propose a new class of Finsler spaces generalized R^h-recurrent spaces characterized by a condition involving the h-covariant derivative of the Cartan's third curvature tensor.

This condition introduces three distinct covariant vector fields that modulate the recurrence behavior, thereby generalizing existing definitions. By exploring transvections, contractions, and covariant derivatives under this framework, we derive a sequence of equivalent forms, each revealing structural aspects of the geometry.

The paper is organized as follows: In Section 2, we define the generalized R^h-recurrent condition and derive its equivalent formulations. Section 3 focuses on deriving key identities and presenting a series of theorems that govern the behavior of curvature and torsion tensors under this generalized recurrence. These results not only reinforce the internal consistency of the defined structure but also highlight novel relations absent in classical settings.

Through this work, we aim to contribute to the deeper understanding of curvature structures in Finsler spaces and to open potential pathways for further generalizations and applications in geometric analysis and theoretical physics.

The theory of recurrence in differential geometry plays a central role in understanding the intrinsic structures of manifolds, particularly within the context of Finsler geometry. Over the past few decades, numerous researchers have contributed to the classification and analysis of recurrent and generalized recurrent structures through the study of curvature tensors and covariant derivatives.

Early foundational work on recurrent manifolds was carried out by Dubey and Srivastava [1981], De and Guha [1991], and Matsumoto [1971], who explored various forms of recurrence including h-isotropic and C^h-recurrent conditions. These contributions were further enriched by investigations into higher-order and specialized recurrences by scholars such as Mishra and Lodhi [2008], Pandey et al. [2011], and Misra et al. [2014], all of whom extended recurrence theory to accommodate more intricate geometric and physical interpretations.

In more recent developments, Ahsan and Ali [2014, 2016] focused on properties of curvature tensors in general relativity and their implications in the broader setting of Finsler spaces. Meanwhile, Al-Qashbari and his collaborators have made significant contributions to the field by examining various generalized curvature tensors including Berwald, Cartan, Weyl, and M-projective tensors using higher-order derivatives and Lie derivatives in Finsler manifolds [Al-Qashbari et al., 2017–2025].

Notably, studies such as those by Al-Qashbari et al. [2024, 2025] introduced new types of generalized recurrent Finsler structures through decomposition and transformation of curvature tensors, including the analysis of G^h-covariant derivatives and recurrence of the fifth order. These efforts have not only enriched the classification of Finsler spaces but also expanded the algebraic and geometric tools used to explore their properties.

Building on this substantial body of work, the present study introduces and investigates a new class of Finsler spaces, referred to as generalized R^h-recurrent spaces. Defined through a recurrence condition imposed on Cartan's third curvature tensor and controlled by three non-null covariant vector fields, this class of spaces reveals structural symmetries and invariant identities that generalize classical recurrence conditions. The results obtained contribute to both the theoretical development of Finsler geometry and its potential applications in mathematical physics.

Let us consider an n-dimensional Finsler space equipped with the metric function F satisfying the requisite conditions. Let consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ^{*i}_{jk} and Berwald's connection parameters G^i_{jk}. These are symmetric in their lower indices and positively homogeneous of degree zero in the directional arguments.

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by

(1.1) $$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1\ , & if \quad i = k \ , \\ 0\ , & if \quad i \neq k \ . \end{cases}$$

The vectors y_i and y^i satisfies the following relations

(1.2) a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $g_{ij} = \dot\partial_i\, y_j = \dot\partial_j\, y_i$,

d) $g_{ij}\, y^j = \frac{1}{2}\, \dot\partial_i\, F^2 = F\, \dot\partial_i\, F$ and e) $\dot\partial_j\, y^i = \delta^i_j$.

The tensor C_{ijk} defined by

(1.3) $$C_{ijk} = \frac{1}{2}\, \dot\partial_i\, g_{jk} = \frac{1}{4}\, \dot\partial_i\, \dot\partial_j\, \dot\partial_k\, F^2$$

is known as (h) hv - torsion tensor. It is positively homogeneous of degree -1 in the directional arguments and symmetric in all its indices.

The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor C_{ijk} are related by

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$, b) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$

and c) $C^h_{ik} = g^{hj}\, C_{ijk}$.

The (v) hv-torsion tensor C^h_{ik} is also positively homogeneous of degree -1 in the directional arguments and symmetric in its lower indices.

É. Cartan deduced the h-covariant derivative for an arbitrary vector filed X^i with respect to x^k

(1.5) $$X^i_{|k} = \partial_k\, X^i - \left(\dot\partial_r\, X^i\right) G^r_k + X^r\, \Gamma^{*i}_{rk} \ .$$

The metric tensor g_{ij} and the vector y^i are covariant constant with respect to a above process, i.e.

(1.6) a) $g_{ij|k} = 0$, b) $y^i_{|k} = 0$ and c) $g^{ij}_{|k} = 0$.

The process of h-covariant differentiation with respect to x^k commute with partial differentiation with respect to y^j for arbitrary vector filed X^i , according to

(1.7) $\dot{\partial}_j \left(X^i_{|k}\right) - \left(\dot{\partial}_j X^i\right)_{|k} = X^r\left(\dot{\partial}_j \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk}$, where

(1.8) a) $\dot{\partial}_j \Gamma^{*r}_{hk} = \Gamma^{*r}_{jhk}$, b) $P^i_{kh}\, y^k = 0 = P^i_{kh}\, y^h$ and c) $P^i_{jkh}\, y^j = P^i_{jkh}$.

The tensor P^i_{kh} is called *v(hv) –torsion tensor* and its associate tensor P_{jkh} is given by

(1.9) a) $g_{rj}\, P^r_{kh} = P_{kjh}$.

The associate tensor P_{ijkh} of the (hv)-curvature tensor $P^{\,i}_{jkh}$ is given by

(1.9) b) $g_{ir}\, P^r_{jkh} = P_{ijkh}$.

The quantities H^i_{jkh} and H^i_{kh}form the components of tensors and they called h-curvature tensor of Berwald (Berwald curvature tensor) and h(v)-torsion tensor, respectively, and defined as follow:

(1.10) a) $H^i_{jkh} = \partial_j\, G^i_{kh} + G^r_{kh}\, G^i_{rj} + G^i_{rhj}\, G^r_k - \partial_j\, G^i_{hk} - G^r_{hk}\, G^i_{rj} - G^i_{rkj}\, G^r_h$,

An b) $H^i_{kh} = \partial_h\, G^i_k + G^r_k\, C^i_{rh} - \partial_k\, G^i_h - G^r_h\, C^i_{rk}$.

They are skew-symmetric in their lower indices, i.e. k and h . Also they are positively homogeneous of degree zero and one, respectively in their directional arguments. They are also related by

(1.11) a) $H^i_{jkh}\, y^j = H^i_{kh}$, b) $H^i_{jkh} = \partial_j\, H^i_{kh}$ and c) $H^i_{jk} = \partial_j\, H^i_k$.

These tensors were constructed initially by mean of the tensor H^i_h , called the deviation tensor, given by

(1.12) $H^i_h = 2\, \partial_h\, G^i - \partial_r\, G^i_h\, y^r + 2\, G^i_{hs}\, G^s - G^i_s\, G^s_h$.

The deviation tensor H^i_h is positively homogeneous of degree two in the directional arguments.

In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (1.11) and (1.12), we have the following:

(1.13) a) $H^i_{jk}\, y^j = -\, H^i_{kj}\, y^j = H^i_k$ and b) $y_i\, H^i_j = 0$.

The quantities H^i_{jkh} and H^i_{kh} are satisfies the following

(1.14) a) $H_{ijkh} = g_{jr} H^r_{ihk}$, b) $H_{jk.h} = g_{jr} H^r_{hk}$ and c) $y_i\, H^i_{jk} = 0$.

Cartan's third curvature tensor R^i_{jkh} satisfies the identity known as Bianchi identity

(1.15) a) $R^i_{jkh|s} + R^i_{jsk|h} + R^i_{jhs|k} + \left(R^r_{mhs} P^i_{jkr} + R^r_{mkh} P^i_{jsr} + R^r_{msk} P^i_{jhr}\right) y^m = 0$

and b) $R_{ijhk} + R_{ihkj} + R_{ikjh} + \left(C_{ijs}\, K^s_{rhk} + C_{ihs}\, K^s_{rkj} + C_{iks}\, K^s_{rjh}\right) y^r = 0$,

where c) $P^r_{ijs} = \dot{\partial}_s \Gamma^{*r}_{ij} - C^r_{is|j} + C^r_{im}\, C^m_{js|k}\, y^k$.

The Ricci tensor R_{jk} , the deviation tensor R^r_h and the curvature scalar R of the curvature tensor R^i_{jkh} are given by

(1.16) a) $R^i_{jkh}\, y^j = H^i_{hk} = K^i_{jhk}\, y^j$, b) $R_{ijhk} = g_{rj}\, R^r_{ihk}$,

c) $R_{jkhm}\, y^j = H_{kh.m}$, d) $R^r_{ihk} = g^{jr} R_{ijhk}$ and e) $R^i_{jkh}\, g^{jk} = R^i_h$,

The contracted tensor R_{kh}(Ricci tensor) and R_k (Curvature vector) are also connected by

(1.17) a) $R_{jk}\, y^k = R_j$, b) $R_{jk}\, y^j = H_k$, c) $R^i_{jki} = R_{jk}$ and c) $R^i_i = R$.

Also this tensor satisfies the following relation too

(1.18) a) $R^i_{jkh} = K^i_{jkh} + C^i_{js}\, K^s_{rhk}\, y^r$, b) $R_{ijkh} = K_{ijkh} + C_{ijs}\, H^s_{kh}$,

and c) $R_{jkhm}\, y^j = H_{kh.m}$.

where $R_{ijkh}$is the associate curvature tensor of R^i_{jkh}. Cartan's fourth curvature tensor K^i_{jkh} and its associate curvature tensor of K_{ijkh}satisfy the following known as Bianchi identities

(1.19) a) $K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0$

and b) $K_{jrkh} + K_{hrjk} + K_{krhj} = 0$.

2. Structural Properties of Generalized R^h-Recurrent Finsler Spaces

The study of curvature recurrence in Finsler geometry has long provided deep insights into the underlying structure and symmetries of geometric manifolds. In this paper, we introduce and investigate a new class of Finsler spaces, referred to as generalized R^h-recurrent Finsler spaces. These spaces are characterized by a specific h-covariant derivative condition imposed on Cartan's third curvature tensor R^i_{jkh}, involving three non-vanishing covariant vector fields λ_l , μ_l and γ_ℓ .

We show that such a condition leads to a set of equivalent curvature identities that govern the behavior of important geometric objects such as the torsion tensor, the deviation tensor, the Ricci tensor, and the curvature scalar. Through a series of transvections and tensor contractions, we derive explicit forms for the h-covariant derivatives of these tensors and prove their non-vanishing nature under the defined structure. These results not only generalize previously known types of recurrence in Finsler spaces but also demonstrate a rich geometric structure that could serve as a foundation for further developments in both pure mathematics and mathematical physics. The analysis confirms that the generalized recurrence condition imposed on the curvature tensor preserves non-trivial geometric content, making the GR^h-RF_n spaces an important addition to the classification of Finsler manifolds.

Let us consider a Finsler space F_n whose Cartan's third curvature tensor R^i_{jkh} satisfies the following condition

(2.1) $R^i_{jkh|l} = \lambda_l R^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) + \frac{1}{4}\gamma_\ell(R^i_h\, g_{jk} - R^i_k\, g_{jh})$, $R^i_{jkh} \neq 0$,

where λ_l , μ_l and γ_ℓ are non-null covariant vectors field. We shall call such space as a generalized R^h-recurrent space. We shall denote it briefly by GR^h-RF_n .

Transvecting of (2.1) by the metric tensor g_{ip} , using (1.6a), (1.16b) and in view of (1.1), where we suppose that $g_{ip}\, R^i_k = R_{pk}$, we get

(2.2) $R_{jpkh|l} = \lambda_l\, R_{jpkh} + \mu_l(g_{hp}\, g_{jk} - g_{kp}\, g_{jh}) + \frac{1}{4}\gamma_\ell(R_{ph}\, g_{jk} - R_{pk}\, g_{jh})$.

Conversely, the transvection of the condition (2.2) by the associate tensor g^{ip} of the metric tensor g_{ip} , yields the condition (2.1). Thus, the condition (2.2) is equivalent to the condition (2.1). Therefore a generalized R^h- recurrent space characterized by the condition (2.2).

Consequently, we deduce the following theorem

Theorem 2.1. A Finsler space GR^h-RF_n is fully characterized by the h-covariant derivative condition given in equation (2.2). This condition serves as an equivalent reformulation of the original definition (2.1), thereby establishing it as a defining property of generalized R^h-recurrent Finsler spaces.

Let us consider GR^h-RF_n characterized by the condition (2.2).

Transvecting the condition (2.1) by y^j, using (1.6b), (1.16a) and (1.2a), we get

(2.3) $H^i_{kh|l} = \lambda_l H^i_{kh} + \mu_l(\delta^i_h\, y_k - \delta^i_k\, y_h) + \frac{1}{4}\gamma_\ell(R^i_h\, y_k - R^i_k\, y_h)$.

Further, transvecting (2.3) by y^k, using (1.6b),(1.13a), (1.2b) and in view of (1.1), we get

(2.4) $H^i_{h|l} = \lambda_l H^i_h + \mu_l(\delta^i_h\, F^2 - y_h\, y^i) + \frac{1}{4}\gamma_\ell(R^i_h\, F^2 - R^i_k\, y_h\, y^k)$.

From the preceding derivations, we establish the following result, formulated as Theorem 2.2

Theorem 2.2. In a generalized GR^h-RF_n space, the h-covariant derivatives of the h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h are governed by the differential relations given in equations (2.3) and (2.4), respectively. These conditions highlight the inherent structural recurrence of these tensors in such a Finsler manifold.

Contracting the indices i and h in the condition (2.1), using (1.17c), (1.17d) and (1.1), we get

(2.5) $R_{jk|l} = \lambda_l R_{jk} + (n-1)\,\mu_l\, g_{jk} + \frac{1}{4}\gamma_\ell(R\, g_{jk} - R^i_k\, g_{ji})$.

Transvecting (2.5) by y^k , using (1.6b), (1.17a) and (1.2a), we get

(2.6) $\quad R_{j|l} = \lambda_l\, R_j + (n-1)\,\mu_l\, y_j + \frac{1}{4}\gamma_\ell\big(R\, y_j - R^i_k\, g_{ji}\, y^k\big)$.

Further, transvecting the condition (2.1) by the associate tensor g^{jk} of the metric tensor g_{jk} , using (1.6c), (1.16e) and in view of (1.1), we get

(2.7) $\quad R^i_{h|l} = \lambda_l\, R^i_h + \mu_l(\,n-1)\,\delta^i_h + \frac{1}{4}\gamma_\ell\big(R^i_h - R^i_k\,\delta^k_h\big)$.

Contracting the indices i and h in condition (2.7), using (1.17d) and (1.1), we get

(2.8) $\quad R_{|l} = \lambda_l\, R + n\,(n-1)\,\mu_l + \frac{1}{4}\gamma_\ell\big(\,R - R^i_k\,\delta^k_i\big)$.

The conditions (2.5), (2.6), (2.7) and (2.8), show that the Ricci tensor R_{jk} , the curvature vector R_j , the deviation tensor R^i_h and the curvature scalar R of a generalized R^h-recurrent space cannot vanish, because the vanishing of them imply the vanishing of the covariant vector field μ_l , i.e. $\mu_l = 0$, a contradiction.

From the above analysis, we establish that

Theorem 2.3. In a generalized GR^h-RF_n space, the Ricci tensor R_{jk}, the curvature vector R_j , the deviation tensor R^i_h , and the scalar curvature R are all necessarily non-vanishing. Their non-vanishing nature is an essential consequence of the defining recurrence condition, and any assumption to the contrary would lead to a contradiction with the vector field $\mu_l = 0$.

3. Identities Involving Higher-Order h-Covariant Derivatives and Torsion Structures in Generalized R^h-Recurrent Finsler Spaces

In this section, we derive and examine several fundamental identities involving h-covariant derivatives of curvature tensors and torsion tensors within the framework of generalized R^h-recurrent Finsler spaces, denoted as GR^h-RF_n. These identities reveal deep interconnections among the Cartan connection, the Berwald connection, and their associated torsion structures. The results are obtained through successive transvections, contractions, and applications of key differential geometric relations, yielding concise formulations that further characterize the internal structure of GR^h-RF_n spaces. These findings not only enrich the algebraic structure of such manifolds but also provide a foundation for investigating more complex geometric behaviors in advanced Finsler geometry.

Taking h- covariant differentiation of the formula (1.15b) with respect to x^l in the sense of Cartan and transvecting (1.15b) by the associate tensor g^{jp} of the metric tensor g_{jp} , using (1.6c), (1.16a), (1.16d) and (1.4a), we get

(3.1) $\quad \big(C^p_{is}\, H^s_{hk} + g^{jp} C_{ihs}\, H^s_{kj} + g^{jp} C_{iks}\, H^s_{jh}\big)_{|l} = -R^p_{ihk|l} - g^{jp} R_{ihkj|l} - g^{jp} R_{ikjh|l}$.

Transvecting (3.1) by y^i, using (1.6b), (1.16a), (1.18c), (1.4b) and (1.4a), we get

(3.2) $\quad H^p_{hk|l} = -\big(g^{jp}\, H_{h.kj|l} + g^{jp}\, H_{k.jh|l}\big)$.

Thus, we conclude Theorem 3.1

Theorem 3.1. In a generalized R^h-recurrent Finsler space GR^h-RF_n, the following identities (3.1) and (3.2) hold.

Using (1.16b) and (1.16a) in the identity (1.15b), we get

(3.3) $\quad g_{rj} R^r_{ihk} + g_{rh} R^r_{ikj} + g_{rk} R^r_{ijh} + C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} = 0$.

Now, transvecting (3.3) by y^i, using (1.16a) and (1.4b), we get

$$g_{rj}\, H^r_{hk} + g_{rh}\, H^r_{kj} + g_{rk}\, H^r_{jh} = 0 \;.$$

Transvecting the above equation by y^r , using (1.2a), we get

(3.4) $\quad y_j H^r_{hk} + y_h H^r_{kj} + y_k H^r_{jh} = 0$.

By using (1.14b), the equation (3.4) yields to

(3.5) $\quad H_{h.jk} + H_{k.hj} + H_{j.kh} = 0$.

Transvecting (3.4) by y^h , using (1.13a) and (1.2a), we get .

$$g_{rj}\,H^r_k = g_{rk}\,H^r_j \quad .$$

Transvecting the above equation by y^r , using (1.2a), we get

(3.6) $\quad y_j\,H^r_k = y_k\,H^r_j \quad .$

Consequently, the findings above lead to the formulation of the following theorem

Theorem 3.2. In a generalized R^h-recurrent Finsler space GR^h-RF_n, the torsion tensor H^r_{hk} satisfies the following cyclic and symmetry identities in (3.4), (3.5), and (3.6).

Using (1.16a) in the identity (1.15a), we get

(3.7) $\quad R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + H^s_{kh}\,P^r_{ijs} + H^s_{jk}\,P^r_{ihs} + H^s_{hj}\,P^r_{iks} = 0 \quad .$

In view of the condition (2.1), the identity (3.7), may be written as

(3.8) $$\lambda_h\,R^r_{ijk} + \lambda_k R^r_{ihj} + \lambda_j\,R^r_{ikh} + \mu_h\big(\delta^r_k g_{ij} - \delta^r_j\,g_{ik}\big) + \mu_k\big(\delta^r_j\,g_{ih} - \delta^r_h g_{ij}\big)$$
$$+ \mu_j(\delta^r_h g_{ik} - \delta^r_k g_{ih}) + \tfrac{1}{4}\gamma_h\big(R^r_k\,g_{ij} - R^r_j\,g_{ik}\big) + \tfrac{1}{4}\gamma_k\big(R^r_j\,g_{ih} - R^r_h\,g_{ij}\big)$$
$$+ \tfrac{1}{4}\gamma_j(R^r_h\,g_{ik} - R^r_k\,g_{ih}) + \big(H^s_{kh}\,P^r_{ijs} + H^s_{jk}\,P^r_{ihs} + H^s_{hj} P^r_{iks}\big) = 0 \quad .$$

Transvecting (3.8) by y^i , using (1.16a), (1.2a) and (1.8c), we get

(3.9) $$\lambda_h H^r_{jk} + \lambda_k H^r_{hj} + \lambda_j H^r_{kh} + \mu_h\big(\delta^r_k\,y_j - \delta^r_j\,y_k\big) + \mu_k\big(\delta^r_j\,y_h - \delta^r_h\,y_j\big)$$
$$+ \mu_j(\delta^r_h y_k - \delta^r_k\,y_h) + \tfrac{1}{4}\gamma_h\big(R^r_k\,y_j - R^r_j y_k\big) + \tfrac{1}{4}\gamma_k\big(R^r_j y_h - R^r_h\,y_j\big)$$
$$+ \tfrac{1}{4}\gamma_j(R^r_h\,y_k - R^r_k\,y_h) + \big(H^s_{kh}\,P^r_{js} + H^s_{jk}\,P^r_{hs} + H^s_{hj}\,P^r_{ks}\big) = 0 \quad .$$

Transvecting (3.9) by y^j, using (1.13a), (1.2b), (1.1) and (1.8b), we get

(3.10) $$\lambda_h H^r_k - \lambda_k H^r_h + \lambda\,H^r_{kh} + \mu_h(\,\delta^r_k\,F^2 - y_k\,y^r) + \mu_k(y_h\,y^r - \delta^r_h\,F^2)$$
$$+ \mu\,(\delta^r_h\,y_k - \delta^r_k\,y_h) + \tfrac{1}{4}\gamma_h\big(R^r_k\,F^2 - R^r_j y_k\,y^j\big) + \tfrac{1}{4}\gamma_k\big(R^r_j y_h y^j - R^r_h\,F^2\big)$$
$$+ \tfrac{1}{4}\gamma\,(R^r_h\,y_k - R^r_k\,y_h) + (H^s_k\,P^r_{hs} - H^s_h\,P^r_{ks}) = 0 \quad ,$$

where $\lambda_j\,y^j = \lambda$, $\gamma_j\,y^j = \gamma$ and $\mu_j\,y^j = \mu$.

These derivations lead us to assert the following

Theorem 3.3. In a generalized R^h-recurrent Finsler space GR^h-RF_n, the curvature tensor and the torsion tensor satisfy the following identities (3.9) and (3.10).

Further, transvecting (3.9) and (3.10) by the vector y_r , using (1.14c), (1.1),(1.13b) and (1.2b), such that $y_r \neq 0$, we get

(3.11) $$\gamma_h\big(R^r_k\,y_j - R^r_j y_k\big) + \gamma_k\big(R^r_j y_h - R^r_h\,y_j\big) + \gamma_j\,(R^r_h\,y_k - R^r_k\,y_h)$$
$$+ 4\big(H^s_{kh}\,P^r_{js} + H^s_{jk}\,P^r_{hs} + H^s_{hj}\,P^r_{ks}\big) = 0$$

and

(3.12) $$+ \gamma_h\big(R^r_k\,F^2 - R^r_j y_k\,y^j\big) + \gamma_k\big(R^r_j y_h y^j - R^r_h\,F^2\big)$$
$$+ \gamma\,(R^r_h\,y_k - R^r_k\,y_h) + 4H^s_k\,P^r_{hs} - 4H^s_h\,P^r_{ks} = 0 \quad .$$

Respectively. Thus, we conclude

Theorem 3.4. In the generalized R^h-recurrent Finsler space GR^h-RF_n, the identity given by equation (3.11) holds valid and reflects an intrinsic geometric property of the space.

Theorem 3.5 In the generalized R^h-recurrent Finsler space GR^h-RF_n, the relation expressed in equation (3.12) is satisfied, confirming the compatibility conditions between the curvature tensors and the recurrence structure.

Transvecting (3.8), (3.9) and (3.10) by the metric tensor g_{rm} , using (1.16b), (1.1), (1.9b), (1.14b), (1.9a) and (1.2a), we get

(3.13) $$\lambda_h\,R_{imjk} + \lambda_k\,R_{imhj} + \lambda_j\,R_{imkh} + \mu_h\,\big(g_{km}\,g_{ij} - g_{jm}\,g_{ik}\big)$$
$$+ \mu_k\,\big(g_{jm}\,g_{ih} - g_{hm}\,g_{ij}\big) + \mu_j(g_{hm}\,g_{ik} - g_{km}\,g_{ih})$$
$$+ \tfrac{1}{4}\gamma_h\big(R^r_k\,g_{ij} - R^r_j\,g_{ik}\big)\,g_{rm} + \tfrac{1}{4}\gamma_k\big(R^r_j\,g_{ih} - R^r_h\,g_{ij}\big)\,g_{rm}$$

$$+\frac{1}{4}\gamma_j(R^r_h\, g_{ik} - R^r_k\, g_{ih})\, g_{rm} + \left(H^s_{kh}\, P_{imjs} + H^s_{jk}\, P_{imhs} + H^s_{hj}\, P_{imks}\right) = 0\ ,$$

(3.14) $$\lambda_h\, H_{jm.k} + \lambda_k\, H_{hm.j} + \lambda_j\, H_{km.h} + \frac{1}{4}\gamma_h\left(R^r_k\, y_j - R^r_j\, y_k\right) g_{rm} + \frac{1}{4}\gamma_k\left(R^r_j y_h - R^r_h\, y_j\right) g_{rm} + \frac{1}{4}\gamma_j(R^r_h\, y_k - R^r_k\, y_h)\, g_{rm} + \left(H^s_{kh}\, P_{jms} + H^s_{jk}\, P_{hms} + H^s_{hj}\, P_{kms}\right) = 0 \quad ,$$

and

(3.15) $$g_{rm}\,(\lambda_h\, H^r_k - \lambda_k\, H^r_h + \lambda\, H^r_{kh}) + \mu_h\,(g_{km}\, F^2 - y_k\, y_m) + \mu_k\,(y_h\, y_m - g_{hm}\, F^2) + \mu\,(g_{hm}\, y_k - g_{km}\, y_h) + \frac{1}{4}\gamma_h\left(R^r_k\, F^2 - R^r_j\, y_k\, y^j\right) g_{rm} + \frac{1}{4}\gamma_k\left(R^r_j\, y_h y^j - R^r_h\, F^2\right) g_{rm} + \frac{1}{4}\gamma\,(R^r_h\, y_k - R^r_k\, y_h)\, g_{rm} + (H^s_k\, P_{hms} - H^s_h\, P_{kms}) = 0\ .$$

These derivations lead us to assert the following

Theorem 3.6. In the generalized R^h-recurrent Finsler manifold GR^h-RF_n, the identities represented by equations (3.13), (3.14), and (3.15) are all valid. These relations result from the transvection of previously derived fundamental identities by the metric tensor and illustrate the deep interdependence between the curvature tensors, torsion tensors, and the structure coefficients of the Finsler space.

4. Conclusions

In this paper, we have introduced and systematically studied a new class of Finsler spaces, denoted by GR^h-RF_n, characterized by a generalized recurrence condition involving Cartan's third curvature tensor and three non-zero covariant vector fields. We demonstrated the equivalence between multiple formulations of the generalized recurrence condition through transvection and contraction operations. Our investigation revealed that essential geometric objects such as the Ricci tensor, deviation tensor, curvature vector, and scalar curvature are inherently non-vanishing in this class of spaces, highlighting a rigid and rich structure compared to classical recurrent spaces. Furthermore, we derived and verified a set of fundamental identities involving curvature and torsion tensors that enhance the theoretical foundation of these spaces.

The results not only generalize existing concepts in Finsler geometry but also offer a broader framework for further explorations. Potential future work includes the study of such generalized recurrent structures under different connections, applications in physical models, and the development of invariant properties under geometric transformations.

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الخصائص البنائية والمتطابقات الاساسية للأطر الريمانية المعممة من النوع التكراري- R في الفضاءات الفينسلرية

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الملخص: في هذه الورقة البحثية، نقدم وندرس صنفًا جديدًا من الفضاءات الفينسلرية يُطلق عليه اسم الفضاءات الفينسلرية ذات التكرار العام المعمم من نوع R^h ، ويرمز لها بـ GR^h-RF_n يتم تعريف هذا الصنف من خلال شرط تكرار معمم يُفرض على موتر الانحناء الثالث لكارتان، ويشمل ثلاثة حقول متجهة تباينية غير منعدمة. نستنتج الخصائص الأساسية لهذه الفضاءات، ونثبت تساويها من خلال عدة تطابقات موترية. كما نحلل سلوك بعض الكيانات الهندسية المرتبطة مثل موتر الالتواء من نوع h(v)، وموتر ريشي، والمتجه الانحنائي، وموتر الانحراف، والانحناء القياسي. علاوة على ذلك، نبرهن على عدد من التطابقات غير البديهية التي تتضمن مشتقات تباينية وتقلصات، مما يكشف عن تناظرات داخلية غنية. وتُختتم الدراسة بعدة مبرهنات بنيوية تُوسّع من المفاهيم التقليدية للتكرار في هندسة فينسلر.

الكلمات المفتاحية: هندسة فينسلر، الفضاءات ذات التكرار المعمم، التكرار من نوع R^h ، موتر الانحناء، المشتقات التباينية.