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Generalized βP-Recurrent Finsler Spaces: Characterization, Properties, and Covariant Derivatives of Weyl's Projective Tensors

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Abstract: This paper investigates the properties and behaviour of generalized βP-recurrent Finsler spaces (GβP-RF$_n$) with an emphasis on the covariant derivatives of Weyl's projective tensors, which include torsion, curvature, and deviation tensors. We define a GβP-recurrent space as a Finsler space where the second curvature tensor satisfies a specific recurrence condition involving non-zero covariant vector fields. Through a series of mathematical derivations, the paper explores the equivalence of different characterizations of the GβP-recurrent space, proving that the torsion tensor, its associate tensor, the P-Ricci tensor, and the curvature vector are non-vanishing. Further, we analyze the relationship between the covariant vector fields λ_ℓ , μ_ℓ and γ_ℓ , showing their dependence or independence on the direction argument. The study also includes the application of Berwald's covariant derivative to the projective curvature tensor, concluding that these tensors exhibit generalized recurrence under certain conditions. The results are presented in a series of theorems that contribute to the deeper understanding of the geometric structure of GβP-RF$_n$ spaces.

Keywords: Berwald's second curvature tensor P^i_{jkh} , Weyl's Projective Tensors W^i_{jkh} , GβP-RF$_n$ space, Torsion tensor, Finsler space.

1. Introduction

Finsler geometry, which generalizes Riemannian geometry, has been a subject of extensive study due to its ability to model a broader range of geometric spaces. A key aspect of Finsler geometry involves the curvature properties of spaces, which can be characterized using various curvature tensors. In this paper, we focus on generalized βP-recurrent Finsler spaces (GβP-RF$_n$), a class of Finsler spaces defined by a specific recurrence condition on the second curvature tensor. The condition links the curvature tensor with non-zero covariant vector fields, resulting in a generalized form of recurrence, which is the core of our study.

The motivation behind this work stems from the need to explore the deeper geometric and analytic properties of these spaces, particularly the behavior of Weyl's projective tensors, which are crucial for understanding the curvature, torsion, and deviation in the context of Finsler geometry. By analyzing these tensors and their covariant derivatives, we aim to develop a set of characterization results for the GβP-RF$_n$ spaces, offering new insights into their structure and behavior.

Through a series of detailed calculations, this paper establishes several fundamental theorems concerning the non-vanishing nature of key geometric quantities, the dependence of covariant vector fields on the direction argument, and the recurrence properties of the projective curvature tensors. These findings contribute to the broader understanding of the geometric properties of Finsler spaces and have potential applications in areas such as gravitational theory and advanced differential geometry.

The structure of the paper is as follows: we begin by defining the concept of a generalized βP-recurrent Finsler space and derive the necessary conditions for such a space to exist. We then present a series of theorems that characterize the non-vanishing nature of the torsion tensor, P-Ricci tensor, and curvature vector, as well as the dependence of the covariant vector fields λ_ℓ , μ_ℓ and γ_ℓ on the direction argument. Finally, we discuss the covariant derivatives of Weyl's projective tensors, showing their generalized recurrence under certain conditions.

In the field of Finsler geometry, significant strides have been made over the years in understanding the behaviour and properties of curvature tensors and their recurrence in various spaces. Numerous studies have explored generalized recurrent Finsler spaces, providing valuable insights into their geometrical structures.

Ahsan and Ali (2014) presented an in-depth analysis of the W-curvature tensor, contributing to the broader understanding of curvature in Finsler spaces. This work laid a foundational framework for subsequent studies on recurrent and generalized recurrent Finsler spaces.

Awed (2017) focused on the study of generalized P^h-recurrent Finsler spaces, delving into the recurrence properties of such spaces, a crucial aspect for understanding the stability and behavior of curvature tensors under different conditions. This dissertation provided essential insights into how curvature tensors interact with the structure of Finsler spaces.

AL-Qashbari et al. (2024) further advanced the study of curvature tensors in recurrent Finsler spaces, investigating R-projective curvature tensors. Their work provided a comprehensive understanding of the interrelations of these tensors, which has been crucial for the development of generalized Finsler spaces. In a similar vein, AL-Qashbari, Abdallah, and Al-ssallal (2024) extended these concepts by introducing higher-order generalizations, refining the characterization of recurrent Finsler structures through the use of special curvature tensors.

Another significant contribution came from AL-Qashbari and Al-ssallal (2024), who utilized Berwald's and Cartan's higher-order derivatives to study curvature tensors in Finsler space, offering new methods for analyzing the geometrical properties of these spaces. Similarly, their work on decomposing Weyl's curvature tensor using Berwald's derivatives (AL-Qashbari, Haoues, and Al-ssallal, 2024) provided further understanding of the relationship between curvature tensors and their transformation properties.

The importance of higher-order recurrence and its applications was emphasized in several works, including AL-Qashbari's (2020) exploration of recurrence decompositions in Finsler space and his study of generalized curvature tensors in B-recurrent Finsler spaces. These studies have laid the groundwork for understanding the recurrence behavior of curvature tensors in complex geometrical structures.

Additionally, studies on specific curvature tensors, such as the work by Ghadle et al. (2024) on generalized BP-recurrent and birecurrent Finsler spaces, and Opondo (2021) on projective curvature tensors in bi-recurrent Finsler spaces, have further enriched the existing literature by providing detailed investigations into these specialized spaces. These studies provide a deeper understanding of how different types of curvature behave under transformation, which is essential for the generalization of Finsler spaces. The broader implications of these studies extend to various fields, including differential geometry, applied mathematics, and theoretical physics, where the properties of curvature tensors play a fundamental role in the analysis of spacetime, fluid dynamics, and other complex systems. The continued research on generalized recurrent Finsler spaces is essential for advancing our understanding of geometric structures and their applications in both theoretical and practical contexts.

The metric tensor g_{ij} and the associate metric tensor g^{ij} are covariant constant with respect to h-covariant derivative, i.e.

(1.1) a) $g_{ij|k} = 0$ and b) $g^{ij}_{|k} = 0$.

(1.2) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 & if \quad i = k \ , \\ 0 & if \quad i \neq k \ . \end{cases}$

The covariant derivative of the vectors y^i and y_i , vanish identically, i. e.

(1.3) $\mathcal{B}_k\, y^i = 0$.

(1.4) a) $y_i\, y^j = F^2$ and b) $g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i$.

The vectors y_i and g_{ij} also satisfy the following relation

(1.5) a) $y_i = g_{ij}(x,y)\, y^j$, b) $\mathcal{B}_k\, y^i = 0$ and c) $\mathcal{B}_k g_{ij} = -2\, y^h \mathcal{B}_h\, C_{ijk}$.

The associate curvature tensor P_{ijkh} of the curvature tensor P^i_{jkh} is given by

(1.6) $P_{ijkh} = g_{ir}\, P^r_{jkh}$

The P- Ricci tensor P_{jk} , the curvature scalar P and the deviation tensor P^i_j are given by

(1.7) a) $P^i_{jki} = P_{jk}$, b) $P^i_{ki} = P_k$ and c) $P^i_i = P$.

The curvature tensor P^i_{kh} satisfies the relations

(1.8) $g_{ir}\, P^{i}_{kh} = P_{rkh}$.

The associate tensor P^{r}_{jkh} of the curvature tensor P_{ijkh} is given by

(1.9) $P^{r}_{jkh} = g^{ir} P_{ijkh}$.

The h-covariant derivative, defined above commute with the partial differentiation with respect to y^{j} according to

(1.10) a) $\dot{\partial}_j\left(X^{i}_{\ |k}\right) - \left(\dot{\partial}_j X^{i}\right)_{|k} = X^{r}\left(\dot{\partial}_j \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^{i}\right) P^{r}_{jk}$, where

b) $P^{r}_{jk} = \left(\dot{\partial}_j \Gamma^{*r}_{hk}\right) y^{h} = \Gamma^{*r}_{jhk}\, y^{h}$.

The hv-curvature tensor P^{i}_{jkh} is positively homogeneous of degree zero in y^{i} and satisfies the relations

(1.11) $P^{i}_{jkh}\, y^{j} = \Gamma^{*i}_{hjk} y^{j} = P^{i}_{kh} = C^{i}_{kh|r} y^{r}$,

(1.12) $P^{i}_{jk}\, y^{j} = 0$.

The tensor $P_{ij} - P_{ji}$ is given by

(1.13) $P_{ijkh}\, g^{kh} = P_{ij} - P_{ji}$.

Due to homogeneity of Γ^{*i}_{jk} in y^{i} the connection parameter Γ^{*i}_{jk} satisfy

(1.14) $\dot{\partial}_h \Gamma^{*i}_{jk}\, y^{k} = 0$.

The projective curvature tensor W^{i}_{jkh} is known as (Wely's projective curvature tensor), the projective torsion tensor W^{i}_{jk} is known as (Wely's torsion tensor) and the projective deviation tensor W^{i}_{j} is known as (Wely's deviation tensor) are defined by

(1.15) $W^{i}_{jkh} = H^{i}_{jkh} + \frac{2\,\delta^{i}_{j}}{(n+1)} H_{[hk]} + \frac{2\,y^{i}}{(n+1)} \dot{\partial}_j H_{[kh]} + \frac{\delta^{i}_{k}}{(n^2-1)}\left(n\, H_{jh} + H_{hj} + y^{r}\dot{\partial}_j H_{hr}\right)$
$- \frac{\delta^{i}_{h}}{(n^2-1)}\left(n\, H_{jk} + H_{kj} + y^{r}\dot{\partial}_j H_{kr}\right)$,

(1.16) $W^{i}_{jk} = H^{i}_{jk} + \frac{y^{i}}{(n+1)} H_{[jk]} + 2\left\{\frac{\delta^{i}_{[j}}{(n^2-1)}\left(n\, H_{k]} - y^{r} H_{k]\,r}\right)\right.$

(1.17) $W^{i}_{j} = H^{i}_{j} - H\,\delta^{i}_{j} - \frac{1}{(n+1)}\left(\dot{\partial}_r H^{r}_{j} - \dot{\partial}_j H\right) y^{i}$, respectively.

The tensors W^{i}_{jkh} , W^{i}_{jk} and W^{i}_{k} are satisfying the following identities

(1.18) a) $W^{i}_{jkh}\, y^{j} = W^{i}_{kh}$ and b) $W^{i}_{jk}\, y^{j} = W^{i}_{k}$.

The projective curvature tensor W^{i}_{jkh} is skew-symmetric in its indices k and h.

The Cartan's third curvature tensor R^{i}_{jkh} , and the R-Ricci tensor R_{jk} in sense of Cartan, respectively, given by

(1.19) a) $R^{i}_{jkh} = \Gamma^{*i}_{hjk} + \left(\Gamma^{*i}_{ljk}\right) G^{l}_{h} + C^{i}_{jm}\left(G^{m}_{kh} - G^{m}_{kl}\, G^{l}_{h}\right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h$,

b) $R^{i}_{jkh}\, y^{j} = H^{i}_{kh}$, c) $R_{jk}\, y^{j} = H_{k}$, d) $R_{jk}\, y^{k} = R_{j}$ and e) $R^{i}_{jki} = R_{jk}$.

Berwald curvature tensor H^i_{jkh} and h(v)-torsion tensor H^i_{kh} form the components of tensors are defined as follow

(1.20) a) $H^i_{jkh} = \partial_j G^i_{kh} + G^r_{kh} G^i_{rj} + G^i_{rhj}\, G^r_k - h/k$

and b) $H^i_{kh} = \partial_h G^i_k + G^r_k\, C^i_{rh} - h/k$.

They are also related by

(1.21) a) $H^i_{jkh}\, y^j = H^i_{kh}$, b) $H^i_{jkh} = \dot{\partial}_j\, H^i_{kh}$ and c) $H^i_{jk} = \dot{\partial}_j\, H^i_k$.

These tensors were constructed initially by means of the tensor H^i_h , called the deviation tensor, given by

(1.22) a) $H^i_h = 2\partial_h\, G^i - \partial_r G^i_h\, y^r + 2\, G^i_{hs}\, G^s - G^i_s\, G^s_h$, where b) $\dot{\partial}_k G^i_h = G^i_{kh}$.

In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (1.21) and (1.22), we have the following:

(1.23) a) $H^i_{jk}\, y^j = H^i_k$, b) $g_{ip}\, H^i_{jk} = H_{jp.k}$ and c) $H_i\, y^i = (n-1)H$.

2. A Study on the Properties and Behavior of Weyl's Projective Tensors in GβP-Recurrent Finsler Spaces

In recent years, the study of Finsler geometry has seen substantial developments, particularly in understanding the behavior and properties of curvature tensors and their recurrence in generalized Finsler spaces. One significant area of research focuses on Weyl's projective tensors, which play a crucial role in describing the curvature structure of Finsler spaces. These tensors are instrumental in analyzing the geometrical properties of GβP-recurrent Finsler spaces, which represent a generalized class of spaces characterized by specific recurrence conditions.

Recent works, including studies by Ahsan and Ali (2014), Awed (2017), and AL-Qashbari et al. (2024), have highlighted the importance of projective tensors and their interactions with curvature tensors in recurrent Finsler spaces. Specifically, these studies explore the recurrence behavior of these tensors and their dependence on various geometric parameters. Additionally, AL-Qashbari's investigations on the higher-order derivatives of curvature tensors, as well as decompositions of Weyl's curvature tensor, provide valuable insights into the structural complexity of such spaces.

This paper aims to contribute to the existing literature by providing a detailed analysis of Weyl's projective tensors in the context of GβP-recurrent Finsler spaces, emphasizing their recurrence properties and the role they play in the overall curvature behavior of these spaces. Through this study, we seek to deepen the understanding of how these tensors influence the geometry of generalized Finsler spaces and their broader applications in differential geometry and theoretical physics. Let us consider an n-dimensional Finsler space F_n

Definition 2.1. A Finsler space F_n whose Berwald's second curvature tensor, P^i_{jkh} satisfies the following condition.

(2.1) $\beta_\ell\, P^i_{jkh} = \lambda_\ell P^i_{jkh} + \mu_\ell\big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\big) + \frac{1}{4}\gamma_\ell\big(P^i_h\, g_{jk} - P^i_k\, g_{jh}\big)$, $P^i_{jkh} \neq 0$

where λ_ℓ , μ_ℓ and γ_ℓ are non-zero covariant vector fields. We shall called it a generalized βP-recurrent space and the tensor whch satisfies the above condition will be called a generalized recurrent tensor .

We shall denote such space briefly by GβP- RF_n .

Remark 2.1. The condition $\beta_\ell\, P^i_{jkh} = \lambda_\ell P^i_{jkh}$, $P^i_{jkh} \neq 0$, looks as a particular case of the condition (2.1) when $\mu_\ell = 0$.

Now, let us consider a Finsler space F_n which is GβP- RF_n .

Transvecting the condition (2.1) by g_{im} , using (1.1a) and (1.6), we get

(2.2) $\beta_\ell\, P_{jmkh} = \lambda_\ell P_{jmkh} + \mu_\ell\big(g_{mj} g_{kh} - g_{mk} g_{jh}\big) + \frac{1}{4}\gamma_\ell\big(P^i_h\, g_{jk} - P^i_k\, g_{jh}\big) g_{im}$.

Conversely, by using (1.1b), (1.9) and (1.2), the tranvection of the condition (2.2) by g^{im} yields (2.1). Thus the condition (2.1) is equivalent to the condition (2.2). Therefore GβP- RF_n may be characterized by the condition (2.2).

Theorem 2.1. The space $G\beta P\text{-}RF_n$ may be characterized by the condition (2.2).

Transvecting the condition (2.1) by y^j , using (1.3), (1.11) and (1.5), we get

(2.3) $\beta_\ell\, P^i_{kh} = \lambda_\ell P^i_{kh} + \mu_\ell\big(\delta^i_k\, y_k - \delta^i_h\, y_h\big) + \frac{1}{4}\gamma_\ell\big(P^i_h\, y_k - P^i_k\, y_h\big)$.

Further, transvecting the condition (2.3) by g_{ij} , using (1.1a) and (1.8), we get

(2.4) $\beta_\ell\, P_{jkh} = \lambda_\ell\, P_{jkh} + \mu_\ell\big(\, g_{jh}\, y_k - g_{jk}\, y_h\,\big) + \frac{1}{4}\gamma_\ell\big(P^i_h\, y_k - P^i_k\, y_h\big)\, g_{ij}$.

Contracting the indices i and h in the condition (2.1), using (1.7a), we get

(2.5) $\beta_\ell\, P_{jk} = \lambda_\ell P_{jk} + (n-1)\,\mu_\ell\, g_{jk} + \frac{1}{4}\gamma_\ell\big(P^i_h\, y_k - P^i_k\, y_h\big)\, g_{ij}$.

Contracting the indices i and h in the condition (2.3) using (1.7b) and (1.7c), we get

(2.6) $\beta_\ell\, P_k = \lambda_\ell\, P_k + (n-1)\,\mu_\ell\, y_k + \frac{1}{4}\gamma_\ell\big(P\, y_k - P^i_k\, y_i\big)$.

The equations (2.3), (2.4), (2.5) and (2.6) show that the (v)hv-torsion tensor P^i_{kh} , its associate tensor P_{jkh} , the P-Ricci tensor P_{jk} and the curvature vector P_k , respectively, can't vanish. Since the vanishing of any one of them would imply the vanishing of the covariant vector field μ_ℓ i.e. $\mu_\ell = 0$, contradiction.

Therefore, we can conclude the following theorem

Theorem 2.2. In the context of GβP-RF_n , the (v)hv-torsion tensor P^i_{kh} , its associated tensor P_{jkh} , the P-Ricci tensor P_{jk}, and the curvature vector P_k are all non-vanishing.

By transvecting condition (2.2) with g^{kh}, and utilizing equations (1.1b), (1.2), and (1.13), we obtain the following result:

(2.7) $\beta_\ell\big(P_{jm} - P_{mj}\big) = \lambda_\ell\big(P_{jm} - P_{mj}\big)$.

Therefore, we can conclude the following theorem

Theorem 2.3. In the context of $G\beta P\text{-}RF_n$, the tensor $(P_{jm} - P_{mj})$ exhibits recurrent behaviour.

By partially differentiating equation (2.6) with respect to y^j and applying equation (1.4b), we obtain the following result.

(2.8) $$\partial_j(\beta_\ell P_k) = (\partial_j\lambda_\ell)P_k + \lambda_\ell(\partial_j P_k) + (n-1)(\partial_j\mu_\ell)\, y_k + (n-1)\,\mu_\ell\, g_{jk}$$
$$+\tfrac{1}{4}(\partial_j\gamma_\ell)(P\, y_k - P_k^i\, y_i) + \tfrac{1}{4}\gamma_\ell(P\, g_{jk} - P_k^i\, g_{ji})\,.$$

Utilizing the commutative formula presented in equation (1.10a) for the tensor P_k, we derive the expression given below.

$$\partial_j(\beta_\ell P_k) - \beta_\ell(\partial_j P_k) = P_r\partial_j\Gamma^{*r}_{k\ell} - (\partial_r P_k)\, P^r_{j\ell}$$
$$+\tfrac{1}{4}(\partial_j\gamma_\ell)(P\, y_k - P_k^i\, y_i) + \tfrac{1}{4}\gamma_\ell(P\, g_{jk} - P_k^i\, g_{ji})\,.$$

Substituting equation (2.8) into this result, we arrive at the subsequent equation.

(2.9) $$\beta_\ell(\partial_j P_k) - P_r\partial_j\Gamma^{*r}_{k\ell} - (\partial_r P_k)\, P^r_{j\ell} = (\partial_j\,\lambda_\ell)P_k + \lambda_\ell\,\partial_j P_k + (n-1)(\partial_j\mu_\ell)\, y_k$$
$$+(n-1)\mu_\ell\, g_{jk} + \tfrac{1}{4}(\partial_j\gamma_\ell)(P\, y_k - P_k^i\, y_i) + \tfrac{1}{4}\gamma_\ell(P\, g_{jk} - P_k^i\, g_{ji})\,.$$

If the tensor $\partial_j P_k$ satisfies the condition:

(2.10) $$\beta_\ell(\partial_j P_k) = (\partial_j\,\lambda_\ell)P_k + (n-1)\,\mu_\ell\, g_{jk}\,.$$

then, by applying equation (2.10) into equation (2.9), we deduce that the non-zero covariant vector field μ_ℓ is dependent on the direction argument if and only if the following condition holds:

(2.11) $$(\partial_j\,\lambda_\ell)P_k + (n-1)(\partial_j\mu_\ell)\, y_k = -\,P_r\partial_j\,\Gamma^{*i}_{k\ell} - (\partial_r P_k)\, P^r_{j\ell} + (n-1)\mu_\ell\, g_{jk}$$
$$+\tfrac{1}{4}(\partial_j\gamma_\ell)(P\, y_k - P_k^i\, y_i) + \tfrac{1}{4}\gamma_\ell(P\, g_{jk} - P_k^i\, g_{ji})\,.$$

Transvecting the condition (2.11) by y^k , using (1.4a), (1.10a) and (1.4b), we get

(2.12) $$(\partial_j\,\lambda_\ell)P_k\, y^k + (n-1)(\partial_j\mu_\ell)\, F^2 = -\,\partial_r(P_k\, y^k)\, P^r_{j\ell} + (n-1)\mu_\ell\, y_j$$
$$+\tfrac{1}{4}(\partial_j\gamma_\ell)(P\, F^2 - P_k^i\, y^k y_i) + \tfrac{1}{4}\gamma_\ell(P\, y_j - P_k^i\, y^k g_{ji})\,.$$

Now, if $P_k\, y^k = 0$, then the equation (2.12), becomes as

$$(n-1)(\partial_j\mu_\ell)\, F^2 + (n-1)\mu_\ell\, y_j + \tfrac{1}{4}(\partial_j\gamma_\ell)(P\, F^2) + \tfrac{1}{4}\gamma_\ell(P y_j) = 0\,.$$

In the above equation, we assume ($n \neq 1$ and $F \neq 0$), then $\partial_j\mu_\ell = -\frac{\mu_\ell\, y_j}{F^2}$, i.e. the non-zero covariant vector field μ_ℓ is dependent of the direction argument, if and only if

$(\partial_j\gamma_\ell)(PF^2) = \gamma_\ell(Py_j)$.

Thus, we may conclude

Theorem 2.4. In the context of $G\beta P\text{-}RF_n$, the non-zero covariant vector field μ_ℓ is direction-dependent on y^j if and only if the condition $(\partial_j\gamma_\ell)(PF^2) = \gamma_\ell(Py_j)$ holds, provided that $P_k\, y^k = 0$ and equation (2.10) is satisfied.

On the other hand, if $P_k\, y^k \neq 0$, then equation (2.12) reveals the following relationship:

(2.13) $(\partial_j\, \lambda_\ell)P_k\, y^k + (n-1)(\partial_j \mu_\ell)\, F^2 = 0$

if and only if

(2.14) $-\partial_r(P_k\, y^k)P^r_{j\ell} + (n-1)\mu_\ell\, y_j + \frac{1}{4}(\partial_j\gamma_\ell)(P\, F^2 - P^i_k\, y^k y_i) + \frac{1}{4}\gamma_\ell(Py_j - P^i_k\, y^k g_{ji}) = 0$.

If (2.14) holds, then the equation (2.13), implies that

(2.15) $\partial_j\mu_\ell = -\{\, (\partial_j\, \lambda_\ell)\, P_k\, y^k \,\} / \,(\, n-1)F^2\}$, where $(\, P_k\, y^k \neq 0\,,\; n \neq 1$ and $F \neq 0\,)$.

by using (2.15), we have if $\partial_j\lambda_\ell = 0$, then $\partial_j\mu_\ell = 0$. Conversely, if $\partial_j\mu_\ell = 0$, then $\partial_j\lambda_\ell = 0$ since $(\, P_k\, y^k \neq 0\,,\; n \neq 1$ and $F \neq 0\,)$.

Thus, we can conclude the following:

Theorem 2.5. In the context of $G\beta P\text{-}RF_n$, the covariant vector field μ_ℓ is independent of the direction argument y^j if and only if the covariant vector field λ_ℓ is independent of the direction argument y^j [provided $P_k\, y^k \neq 0$, (2.10) and (2.14) hold].

Transvecting the condition (2.9) by y^j, using Euler's theorem on homogeneous functions (1.3), (1.14), (1.12) and (1.5), we get

(2.16) $(\partial_j\, \lambda_\ell)P_k\, y^j + (n-1)(\partial_j\mu_\ell)\, y_k y^j + (n-1)\, \mu_\ell\, y_k = 0$

$\quad + \frac{1}{4}(\partial_j\gamma_\ell)(P\, y_k - P^i_k\, y_i)y^j + \frac{1}{4}\gamma_\ell(P\, y_k - P^i_k\, y_i)$.

Now , if $(\partial_j\, \lambda_\ell)\, y^j = 0$, then from (2.16) , we get

(2.17) $(n-1)(\partial_j\mu_\ell)y_k\, y^j + (n-1)\mu_\ell\, y_k + \frac{1}{4}(\partial_j\gamma_\ell)(Py_k - P^i_k\, y_i)y^j + \frac{1}{4}\gamma_\ell(Py_k - P^i_k\, y_i) = 0$.

Spouse $n = 4$, then the equation (2.17), implies that

$$(\partial_j\mu_\ell)\, y_k y^j = -\,\mu_\ell\, y_k + \frac{1}{12}(\partial_j\gamma_\ell)(Py_k - P^i_k\, y_i)y^j + \frac{1}{12}\gamma_\ell(P\, y_k - P^i_k\, y_i)\, .$$

Thus, we can conclude the following:

Theorem 2.6. In the context of $G\beta P\text{-}RF_n$, the following equation holds:

$$(\partial_j\mu_\ell)\, y_k y^j = -\,\mu_\ell\, y_k + \frac{1}{12}(\partial_j\gamma_\ell)(Py_k - P^i_k\, y_i)y^j + \frac{1}{12}\gamma_\ell(P\, y_k - P^i_k\, y_i)\, ,$$

if and only if the condition $(\partial_j\, \lambda_\ell)\, y^j = 0$ is satisfied.

For a Riemannian space V_4 , the projective curvature tensor P^i_{jkh} (Cartan's second curvature tensor) and the divergence of the W-tensor, in terms of the divergence of the projective curvature tensor, can be expressed as follows:

(2.18) $W^i_{jkh} = P^i_{jkh} + \frac{1}{3}\,(\, \delta^i_k\, R_{jh} - R^i_h\, g_{jk}\,)$.

Taking covariant derivative of first order (Berwald's covariant differential operator) of (2.18) with respect to x^m , we get

(2.19) $\mathcal{B}_m W^i_{jkh} = \mathcal{B}_m P^i_{jkh} + \frac{1}{3}\,\mathcal{B}_m(\, \delta^i_k\, R_{jh} - R^i_h\, g_{jk}\,)$

Using the condition (2.1) in (2.19), we get

$$\mathcal{B}_m W^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}\gamma_m(P^i_h g_{jk} - P^i_k g_{jh})$$
$$+\tfrac{1}{3}\mathcal{B}_m(\delta^i_k R_{jh} - R^i_h g_{jk}) .$$

In view of equation (2.18) and by using (1.5c), the above equation can be written as

(2.20) $$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}\gamma_m(P^i_h g_{jk} - P^i_k g_{jh})$$
$$-\tfrac{1}{3}\lambda_m(\delta^i_k R_{jh} - R^i_h g_{jk}) + \tfrac{1}{3}\delta^i_k \mathcal{B}_m R_{jh} - \tfrac{1}{3}(\mathcal{B}_m R^i_h) g_{jk} + \tfrac{2}{3} R^i_h\, y^n \mathcal{B}_n C_{jkm} .$$

This, shows that

$$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}\gamma_m(P^i_h g_{jk} - P^i_k g_{jh}) ,$$

if and only if

(2.21) $$\delta^i_k \mathcal{B}_m R_{jh} - \lambda_m(\delta^i_k R_{jh} - R^i_h g_{jk}) - (\mathcal{B}_m R^i_h) g_{jk} + 2 R^i_h\, y^n \mathcal{B}_n C_{jkm} = 0 .$$

Therefore, using the above assumptions and mathematical analysis results the following

Thus, we can conclude the following:

Theorem 2.7. In the context of $G\beta P\text{-}RF_n$, (for $n = 4$), Berwald's covariant derivative of the first order for the Weyl's projective curvature tensor W^i_{jkh} is generalized recurrent if and only if the condition (2.21) holds.

Transvecting (2.20) by y^j , using (1.5c), (1.18a), (1.1a), (1.19c) and (1.4c), yields

(2.22) $$\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{4}\gamma_m(P^i_h y_k - P^i_k y_h)$$
$$-\tfrac{1}{3}\lambda_m(\delta^i_k H_h - R^i_h H_k) + \tfrac{1}{3}\delta^i_k \mathcal{B}_m H_h - \tfrac{1}{3}(\mathcal{B}_m R^i_h) y_k .$$

This, shows that

(2.23) $$\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{4}\gamma_m(P^i_h y_k - P^i_k y_h)$$

if and only if

(2.24) $$\delta^i_k \mathcal{B}_m H_h - \lambda_m(\delta^i_k H_h - R^i_h H_k) - (\mathcal{B}_m R^i_h) y_k = 0 .$$

Therefore, it is concluded the following theorem

Theorem 2.8. In the context of $G\beta P\text{-}RF_n$, (for $n = 4$), Berwald's covariant derivative of the first order for Weyl's projective torsion tensor W^i_{kh} is given by the equation (2.23) if and only if (2.24) holds.

Transvecting (2.22) by y^k , using (1.5c), (1.18b), (1.1b), (1.2) and (1.23c), we get

$$\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m(\delta^i_h F^2 - y_h y^i) + \tfrac{1}{4}\gamma_m(P^i_h F^2 - P^i_k y_h y^k)$$
$$-\tfrac{1}{3}\lambda_m(H_h y^i - (n-1) R^i_h H) + \tfrac{1}{3} y^i \mathcal{B}_m H_h - \tfrac{1}{3}(\mathcal{B}_m R^i_h) F^2 .$$

This, shows that

(2.25) $$\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m(\delta^i_h F^2 - y_h y^i) + \tfrac{1}{4}\gamma_m(P^i_h F^2 - P^i_k y_h y^k) ,$$

if and only if

(2.26) $\quad y^i\, \mathcal{B}_m\, H_h - \lambda_m\big(H_h\, y^i - (n-1)\, R^i_h\, H\big) - \big(\,\mathcal{B}_m R^i_h\,\big)\, F^2 = 0$.

Thus, the following is derived.

Theorem 2.9. In the context of $G\beta P\text{-}RF_n$, (for $n = 4$), Berwald's covariant derivative of the first order for Weyl's projective deviation tensor W^i_{kh} is given by the equation (2.25) if and only if (2.26) holds.

3. Conclusions

In this study, we thoroughly investigated the properties of generalized βP-recurrent Finsler spaces ($G\beta P\text{-}RF_n$). The analysis focused on the covariant derivatives of Weyl's projective tensors, which are composed of the torsion tensor, curvature tensor, and deviation tensor.

Key conclusions include:

1. **Non-Vanishing Tensors:** In $G\beta P\text{-}RF_n$ spaces, the torsion tensor, its associated tensor, the P-Ricci tensor, and the curvature vector cannot vanish. The non-vanishing nature of these tensors is essential for the recurrence behavior of the space.
2. **Characterization of the GβP-RF_n Space:** The space $G\beta P\text{-}RF_n$ can be characterized by specific recurrence conditions on the curvature tensors and the associated covariant vector fields. The equivalence between different characterizations (such as conditions (2.1) and (2.2)) further solidifies the understanding of this geometric structure.
3. **Dependence of Covariant Vector Fields:** We demonstrated that the non-zero covariant vector field μ_ℓ depends on the direction argument y^j under certain conditions. Specifically, μ_ℓ is direction-dependent if and only if specific conditions involving the covariant derivative of γ_ℓ and the curvature tensors are satisfied.
4. **Recurrent Behavior of the Tensors:** The study shows that the transvection conditions for the curvature tensors lead to recurrent behaviors for both the projective curvature and torsion tensors under specific conditions.
5. **Berwald's Covariant Derivative:** The application of Berwald's covariant derivative to Weyl's projective tensors reveals their generalized recurrence under certain conditions. The analysis of the covariant derivatives also provides insight into how these tensors behave under transformation, contributing to a deeper understanding of the geometric properties of $G\beta P\text{-}RF_n$ spaces.
6. **Mathematical Theorems:** Several theorems were derived to support these findings, each contributing to the foundational understanding of the geometric structure and recurrence behavior of these spaces. These results not only provide valuable theoretical insights but also pave the way for future applications in advanced geometric and topological studies.

In conclusion, this paper establishes the crucial properties and recurrence conditions for generalized βP-recurrent Finsler spaces, offering a robust framework for understanding the geometric structure and behavior of these spaces. The results can be extended to further studies in differential geometry and applied mathematical physics.

4. Recommendations

The authors emphasize the importance of ongoing research and development in Finsler geometry, given its significant potential applications across various scientific disciplines. Further investigation into its properties and structures is crucial for advancing both theoretical understanding and practical implementations in related fields.

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فضاءات فينسلر المعممة -βP : الوصف، الخصائص، والمشتقات التوافقية للموترات الإسقاطية لويلي

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الملخص: تتناول هذه الورقة البحثية خصائص وسلوك الفضاءات الفينسلرية العامة - βP التكرارية ($\mathrm{G\beta P\text{-}RF_n}$) مع التركيز على المشتقات التوافقية لأوتار المشروع لويلي، التي تشمل أوتار التواء، وانحناء، وانحراف. نعرف الفضاء التكراري GβP كفضاء فينسلري حيث يحقق الموتر الثاني للانحناء شرط تكراري محدد يتضمن مجالات متجهة تغايرية غير صفرية. من خلال سلسلة من الاستنتاجات الرياضية، تستكشف الورقة تساوي التوصيفات المختلفة للفضاء التكراريGβP ، حيث تثبت أن موتر التواءه، موتره المساعد، موتر P- ريشي ، والمتجه المنحني جميعها غير منعدمة. علاوة على ذلك، نحلل العلاقة بين المجالات المتجهة التغايرية λ_ℓ ، μ_ℓ ، γ_ℓ ، مظهرين اعتمادها أو استقلالها عن اتجاه المتغير. تشمل الدراسة أيضًا تطبيق المشتقة التغايرية لبروالد على موتر الانحناء الاسقاطي، وتصل إلى استنتاج أن هذه الأوتار تظهر تكرارًا عامًا تحت شروط معينة. تُعرض النتائج في سلسلة من النظريات التي تسهم في الفهم الأعمق للبنية الهندسية لفضاءات $\mathrm{G\beta P\text{-}RF_n}$.

كلمات مفتاحيه: موتر الانحناء الثاني لبروالد P^i_{jkh} ، الموتر الإسقاطي لويلي W^i_{jkh} ، فضاء $\mathrm{G\beta P\text{-}RF_n}$ ، موتر التواء ، فضاء فينسلر.